Inverse Problems and Signal De-noising

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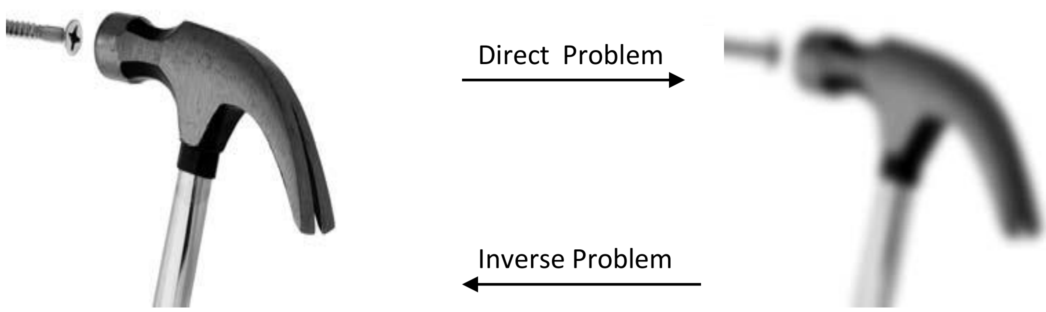
Abstract **-** Inverse problem is a vibrating and emerging topic of mathematics widely applied in many fields of engineering and sciences. The inverse problems are ill-posed by nature, hence dealing them is not an easy task. In order to solve an inverse problem, the problem is converted to an well-posed equivalent problem and then it is solved. This process is called regularization. In this paper, different regularization methods are presented.

Keywords - Inverse problems, regularization, well-posedness, total variation regularization, L-curve.

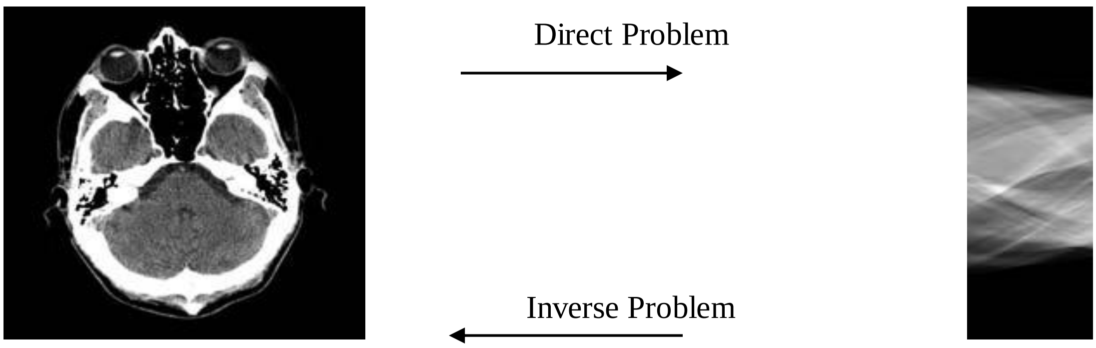
I. INTRODUCTION

Inverse Problems are described as problems where the answer is known, but not the question. Or where the results, or consequences are known, but not the cause. Or where the output is known but not the input. Other form of inverse problem is that wherever are the indirect measurements, there are inverse problems. The interdisciplinary nature of inverse problems makes it applicable in diverse fields of engineering and sciences. As of today, it is being used widely in fields of civil engineering, mechanical engineering, computer science, electronics and communication engineering , physics [1], geophysics[2], chemistry, biotechnology, biomedical engineering, etc. Inverse problems are widely used in signal processing [3], image processing, bio-medical imaging [4, 5], etc. In [1], de-noising of a signal of Raman spectra has been modeled as an inverse problem, then its solution is obtained. Fig. 1 shows the application of inverse problem in image processing. In image processing, the direct problem is as finding out how a given sharp photograph would look like if the camera was incorrectly focused. The inverse problem known as deblurring is finding the sharp photograph from a given blurry image. Here, the cause is the sharp image and the effect is the blurred image. Fig. 2 shows the application of inverse problem in X-ray tomography. In medical X-ray tomography the direct problem would be to find out what kind of X-ray projection images would we get from a patient whose internal organs we know precisely. The corresponding inverse problem is to reconstruct the three-dimensional structure of the patient's insides given a collection of X-ray images taken from different directions.



Source: Internet

Fig. 1: Original and blurred images



Source: Internet

Fig. 2**:** Internal organ and its X-ray image

Here, the patient is the cause and the collection of X-ray images is the effect.

Mathematically, an inverse problem is to obtain *x* for an observed output *y* such that y = Fx (at least approximately), where F is an operator called forward operator. This operator tells unambiguously the association between the data *y* and the model parameters *x* and is demonstration of the physical system from which the inverse problem is generated. Two types of inverse problems are encountered depending on the nature of operator. If F is linear, inverse problem is called linear inverse problem otherwise non-linear inverse problem [6].

Assume, X and Y be two spaces (normally Banach or Hilbert spaces).

$$x \in X \qquad F : X \to Y$$

Direct Problem: Given and, find *Fx* such that

$$Y = Fx \tag{1}$$

Inverse Problem: Given an observed output *y*, find an input *x* that produces it.

$$x \in X \to y = Fx \in Y \tag{2}$$

OR

Given a desired output *z*, find an input *x* that produces an output *y* that is as "close" to *z* as possible:

$$\min_{x \in X} \|Tx - z\|_{l_2}$$

Here, the norm depends on the nature of application. This may be l_1-norm l_0–norm or any other depending on the problem under consideration.

II. ILL-POSEDNESS

The concept of ill-posedness was introduced by Hadamard (1865-1963). A problem is called well-posed provided the three conditions given below are satisfied:

- Existence of Solution : The problem must have a solution for a given suitable data set.
- Uniqueness: The solution must be unique.
- Stability**:** The solution must depend continuously on the observation.

If any of these three conditions are not satisfied, the problem is called as Ill-posed problem.

III. INVERSE PROBLEMS AS IMAGE/SIGNAL DE-NOISING

Inverse problems are mathematically formulated using a data fidelity and a regularizer term as [7]

$$\min_{x \in X} F(x) + R(x),$$

where, *F* is data fidelity term and *R* is regularization term. In imaging problems, the common data fidelity term is taken as

$$F(x) = \frac{1}{2}\|H(x) - y\|^2$$

Here, *H* represents forward operator, y represents the noisy data and ||.|| represents Hilbert norm. Likewise, common regularization term is

$$R(x) = \frac{\lambda}{2}|x|^2$$

where λ is the regularization parameter while | · | represents a Hilbert norm or seminorm depending on the problem under consideration. Many other forms of regularization terms are proposed in literature during last decades. Total variation regularization is one of them which got popularity since its introduction in [8]. The benefits of total variation regularization lies in its characteristics of preservation of corners and edges. The regularization term is considered to be

$$R(x) = \lambda \int_{\Omega} |Dx| \, d\Omega$$

where *x* is defined on the bounded domain $\Omega \subset R^d$. Bounded variation seminorm is considered here.

When the forward operator *H* is taken as identity *I*, the problem is changed into image de-noising. The fidelity term is converted in the form

$$F(x) = \frac{1}{2}\|x - y\|^2$$

In de-noising a signal or image, different regularization terms in total variation de-noising are taken according to the problem under consideration.

IV. REGULARIZATION

By nature, most of the inverse problems are ill-posed and the third condition is normally violated. Hence, finding their solution is a difficult task. Well-posed problems can be solved on computers employing a suitable stable algorithm, but solving an ill-posed problem is complex. To triumph over this difficulty, the problem is re-formulated and for numerical treatment of this problem, some assumptions are incorporated. The process of incorporating additional assumptions is celebrated as regularization.

The first method for solving linear ill-posed problem was proposed by Tikhonov and Arsenin [9] which emerged as a basic method in applied Mathematics for the development of numerous methods in different fields of engineering and sciences. Tikhonov regularization is very frequently used method for regularization of linear ill-posed problems. Solution of inverse problem is found by addition of a stabilization term in minimization problem. Later on a modified version of Tikhonov regularization was also proposed [10]. Other methods are discussed in [11, 12, 13, 14].

Equation $Fx = y$ is re-formulated as a minimizing problem

$$J(x) = \|Fx - y\|_2^2, \; x \in X$$

A stabilization is given by adding a term to the functional

$$J(x) := \|Fx - z\|_2^2 + \lambda \|Lx\|^2, \; x \in X$$

Minimization of above functional is equivalent to finding x* such that ,

$$x^* = \arg\min_{x \in X} \left\{ \|Fx - z\|_2^2 + \lambda \|Lx\|_2^2 \right\}$$

Parameter λ is called the regularization parameter, L is called the regularization function and the incorporated term $\|Lx\|_2^2$ is called the regularizer or the penalty function.

Solving non-linear ill-posed problems is a difficult task. A lot of work is supposed yet to be done for solving a non-linear ill-posed problem. Several techniques have been proposed last two decades [10, 15, 16, 17].

V. CONCLUSION

Signal de-noising is considered as an inverse problem. This inverse problem is converted into optimization problem. Available techniques can be used to solve this optimization problem. The results give the de-noised signals.

VI. REFERENCES

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