Math 3331 - ODEs

We now consider second order ODE's (Linear) where we are given 1 sol^n

<u>Ex</u> $y'' - 2y' + y = 0 \quad y_1 = e^x$

The procedure is to seek a second sol^n of the form $y = ue^x$ & find $u = u(x)$

So $y' = u'e^x + ue^x$

$y'' = u''e^x + u'e^x + u'e^x + ue^x$

<u>sub</u> $y'' - 2y' + y = 0$

$u''e^x + 2u'e^x + ue^x - 2(u'e^x + ue^x) + ue^x = 0$

$u''e^x + \cancel{2u'e^x} + ue^x - \cancel{2u'e^x} - 2ue^x + ue^x = 0$

$u''e^x = 0 \Rightarrow u'' = 0 \quad u' = c_1 \quad u = c_1x + c_2$

$y = (c_1x + c_2)e^x = c_1xe^x + c_2e^x$

Every 2^{nd} order ODE has 2 independent sol^n (we'll define independent in a minute)

so $y_1 = e^x$, $y_2 = x e^x$

ex2 $\quad x^2 y'' - xy' + y = 0 \qquad y_1 = x$

let $y = xu$, $y' = xu' + u$, $y'' = xu'' + 2u'$

sub $\quad x^2(xu'' + 2u') - x(xu' + u) + xu = 0$

$x^3 u'' + 2x^2 u' - x^2 u' \cancel{- xu + xu} = 0$

$\Rightarrow \quad x^3 u'' + x^2 u' = 0$

a $\quad u'' + \dfrac{u'}{x} = 0 \leftarrow$ from Monday

let $v = u'$ so $v' = u''$ so $v' + \dfrac{v}{x} = 0$ Sep

$\dfrac{dv}{v} = -\dfrac{dx}{x} \Rightarrow \ln|v| = -\ln|x| + \ln c_1$

$v = \dfrac{c_1}{x} \quad u' = \dfrac{c_1}{x} \qquad u = c_1 \ln|x| + c_2$

$y_2 = (c_1 \ln|x| + c_2) e^x$

and sol^n

$y = e^x \ln|x|$

Linear independence

Every linear 2^{nd} order ODE

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

has 2 linearly independent sol^{n}s $\{y_1, y_2\}$

and the general sol^n is

$$y = c_1 y_1 + c_2 y_2 \qquad c_1, c_2 \text{ arbitrary const.}$$

So $y'' - 3y' + 2y = 0$

has the 2 sol^n $\{e^x, e^{2x}\}$

check

$y = e^{2x}$ $y' = 2e^{2x}$ $y'' = 4e^{2x}$

$$\begin{aligned} LS &= y'' - 3y' + 2y \\ &= 4e^{2x} - 3(2)e^{2x} + 2e^{2x} \\ &= (4-6+2)e^{2x} \\ &= 0 = RS \checkmark \end{aligned}$$

Linear Independence

if $c_1 y_1 + c_2 y_2 = 0$ iff $c_1 = c_2 = 0$

for all x

<u>Ex</u> if $c_1 e^{x} + c_2 e^{2x} = 0$ for all x

only if $c_1 = c_2 = 0$

Aside if true for all x then true for $x = 0$

$c_1 e^{0} + c_2 e^{0} = 0$ $\quad c_1 + c_2 = 0$

So $c_2 = -c_1$

So $c_1 e^{x} - c_1 e^{2x} = 0$

$c_1 (e^{x} - e^{2x}) = 0$ $\quad c_1 e^{x}(1 - e^{x}) = 0$

if this is true for all x then $c_1 \equiv 0$.

Wronskian

if $c_1 y_1 + c_2 y_2 = 0$
$c_1 y_1' + c_2 y_2' = 0$ } eliminate c_2

$c_1 y_1 y_2' + c_2 y_2 y_2' = 0$ — sub
$c_1 y_1' y_2 + c_2 y_2 y_2' = 0$

$c_1 (y_1 y_2' - y_1' y_2) = 0$

$c_1 \neq 0$ then $y_1 y_2' - y_1' y_2 = 0$ ← Call this the Wronskian

$W = y_1 y_2' - y_2 y_1'$ From linear Algebra

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$ ← a 2x2 matrix and we calculate the determinant

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$$ for linear independence