FACTOR TERNARY SEMIMODULES

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Abstract: This manuscript is a study on factor ternary semimodules over ternary semiring using congruence relations and further the study extended on the notion of cancellative congruence. It is established that, if \( R \) is a ternary semiring, then the subset \( N \) of factor ternary semimodule \( M \) is a subtractive ternary subsemimodule if and only if there exists an \( R \)-homomorphism satisfying \( N = \ker \alpha \).

Keywords: Ternary semiring, Ternary semimodule, Factor ternary semimodule, Congruence relation.


INTRODUCTION:


T.K. Dutta and S. Kar [5, 6] introduced the notions of ternary semiring and ternary semimodules over a ternary semiring. They investigated regular ternary semiring and developed the theory of ideals in ternary semirings and characterized the Jacobson radical of a ternary semiring by using ternary semimodules.

The notion of congruence was first introduced by Karl Fredrich Gauss in the beginning of 19th century. Congruence is a special type of equivalence relation which has a vital role in the study of quotient structures of different algebraic structures. We study the quotient structure of ternary semiring by using the notion of congruence in ternary semiring.

Ternary semimodules over ternary semirings constitute a fairly natural generalization of semimodules over semirings [2]. In this manuscript we introduce the notion of ternary subsemimodule generated in terms of elements.

Congruence relations played an important role in the theory of semirings and ternary semirings. So we would expect them to have a similar role in the theory of ternary semimodules.

The main purpose of this manuscript is to classify congruences on ternary semirings and study some basic properties of congruences on ternary semirings and also introduced the cancellative congruence.

2. PRELIMINARIES:

DEFINITION 2.1[5]. A nonempty set \( S \) together with a binary addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if \( S \) is an additive commutative semigroup satisfying the following conditions:

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(i) \((abc)de = (a(bcd))e = ab(cde)\)

(ii) \((a + b)cd = acd + bcd\)

(iii) \((a(b + c))d = abd + acd\)

(iv) \(ab(c + d) = abc + abd\) for all \(a, b, c, d, e \in S\)

**Example 2.1.** [5] Let \(X\) be a topological space and \(R^-\) the set of all negative real numbers. Suppose that \(S = \{f: X \to R^-| f \text{ is a continuous map}\}\). We define addition and multiplication on \(S\) by 
\[
(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fgh)(x) = f(x)g(x)h(x)
\]
for all \(x \in X\) and \(f, g, h \in S\). Then we can easily check that \(S\) forms a ternary semiring.

**Definition 2.2.** [5]: A ternary semiring \(S\) is called a commutative ternary semiring if \(abc = bac = bca\) for all \(a, b, c \in S\).

**Definition 2.3.** [5] An additive subsemigroup \(T\) of \(S\) is called a ternary semiring of \(S\) if \(t_1t_2t_3 \in T\), for all \(t_1, t_2, t_3 \in T\).

If \(A, B, C\) are three subsets of \(S\), then \(ABC = \{\Sigma a_i b_i c_i / a_i \in A, b_i \in B, c_i \in C\}\).

**Definition 2.4.** [5]: 
If \(A, B, C\) are three subsets of \(S\), then \(ABC = \{\Sigma a_i b_i c_i / a_i \in A, b_i \in B, c_i \in C\}\). \(S\) forms a ternary semiring.

**Definition 2.5.** [5]: An element \(a\) in a ternary semiring \(S\) is called regular if there exists an element \(x \in S\) such that \(axa = a\). A ternary semiring is called regular if all of its elements are regular.

**Definition 2.6.** An additive subsemigroup \(M\) with a zero element \(0_M\) is called a left ternary semimodule over a ternary semiring \(S\) if there exists a mapping \(S \times S \times M \to M\) satisfying the following conditions:

(i) \(s_1s_2(m_1 + m_2) = s_1s_2m_1 + s_1s_2m_2\)

(ii) \(s_1(s_2s_3)m_1 = s_1s_2s_3m_1 + s_1s_2m_3\)

(iii) \((s_1 + s_2)s_3m_1 = s_1s_3m_1 + s_2s_3m_1\)

(iv) \(s_1s_2(s_3m_1) = s_1s_2s_3m_1 = (s_1s_2s_3)s_3m_1\)

(v) \(s_1s_20_m = 0_m = s_10_m = 0_s s_2m_1\) for all \(m_1, m_2 \in M\) and for all \(s_1, s_2, s_3, s_4 \in S\).

**Remark 2.1.** In addition to the above conditions if \(\sum e_if_im = m\) holds for all \(m \in M\), where \((e_i, f_i)\) is an identity element of \(S\), then \(M\) is said to be a unitary left ternary \(S\)-semimodule.

**Example 2.2.** Let \(M_2(Z^-)\) be the ternary semiring of all \(2 \times 2\) square matrices over \(Z^-\), the set of all negative integers. Then \(I_2 = \{\left[\begin{array}{cc} a & b \\ 0 & 0 \end{array}\right] : a, b \in Z^-\}\) forms a left ternary semimodule over \(M_2(Z^-)\).

**Definition 2.7.** A nonempty subset \(N\) of a right ternary \(S\)-semimodule \(M\) is said to be a ternary subsemimodule of \(M\) if (i) \(a + b \in N\) and (ii) \(ast \in N\) for all \(a, b \in N\) and \(s, t \in S\).

**Remark 2.2.** It is noted that the ternary subsemimodule \(N\) of a left ternary \(S\)-semi module \(M\) always contains the zero of \(M\).

**Definition 2.8.** Let \(S\) be a ternary semiring and \(M\) be a left ternary \(S\)-semi module. Then an equivalence relation \(\rho\) on \(M\) is said to be an \(S\)-congruence relation if and only if

(i) \(m_\rho m'\) and \(n_\rho n'\) in \(M \Rightarrow (m + n)\rho(m' + n')\)
(ii) \( m \rho m' \Rightarrow s_1 s_2 m \rho s_1 s_2 m' \) for all \( s_1, s_2 \in S \)

We denote the set of all S-congruence relations on \( M \) by \( \text{S-cong}(M) \).

**NOTE 2.1.** An S-congruence relation is a ternary sub semimodule of \( M \times M \).

**DEFINITION 2.9.** An S-congruence relation \( \rho \) on a left ternary S-semimodule defined by \( m \rho m' \Leftrightarrow m = m' \) is called the trivial S-congruence, denoted by \( \equiv_t \) and an S-congruence relation \( \rho \) on a left ternary S-semimodule defined by \( m \rho m' \) for all \( m, m' \in M \) is called the universal S-congruence, denoted by \( \equiv_u \).

**NOTE 2.2.** Since \( \equiv_t, \equiv_u \in S - \text{cong}(M) \) we have \( \text{S-cong}(M) \neq \emptyset \).

**DEFINITION 2.10.** A left ternary S-semimodule \( M \neq \{0\} \) is said to be simple ternary semimodule if and only if \( M \) has only two S-congruence relations \( \equiv_t, \equiv_u \).

**NOTE 2.3.** \( \text{S-cong}(M) \) is a partially ordered by the relation \( \leq \) defined by \( \rho \leq \rho' \Leftrightarrow m \rho m' \Rightarrow m \rho' m' \).

**PROOF:**
(i) Reflexive: Clearly \( m \rho m' \Rightarrow m \rho m' \) for all \( \rho \in \text{S-cong}(M) \)
\[ \Rightarrow \rho \leq \rho \text{ for all } \rho \in \text{S-cong}(M) \).
(ii) Anti-symmetric: Suppose \( \rho \leq \rho' \) and \( \rho' \leq \rho \)
\[ \Rightarrow m \rho m' \Rightarrow m \rho' m' \text{ implies } m \rho m' \]
\[ \Rightarrow (m, m') \in \rho \text{ and only if } (m, m') \in \rho' \Rightarrow \rho = \rho' \]
(iii) Transitive: Suppose \( \rho \leq \rho' \) and \( \rho' \leq \rho'' \)
\[ \Rightarrow m \rho m' \text{ implies that } m \rho' m' \text{ and } m \rho' m' \text{ implies that } m \rho'' m' \]
\[ \Rightarrow m \rho m' \text{ implies that } m \rho'' m' \Rightarrow \rho \leq \rho'' \]
Therefore "\( \leq \)" is a partial order relation on \( \text{S-cong}(M) \).

**NOTE 2.4.** \( \equiv_t \leq \rho \leq \equiv_u \) for all S-congruence relations \( \rho \) in \( \text{S-cong}(M) \)

**PROOF:** Since \( \rho \) is S-congruence for \( (m, m) \in \equiv_t \Rightarrow (m, m) \in \rho \Rightarrow \equiv_t \leq \rho \)

Also since \( \equiv_u \) is universal for \( (m, m') \in \rho \Rightarrow (m, m') \in \equiv_u \Rightarrow \rho \leq \equiv_u \).
Therefore \( \equiv_t \leq \rho \leq \equiv_u \).

**NOTE 2.5.** Let \( W \) be a non empty subset of \( \text{S-cong}(M) \). Define a relation \( \rho \) on \( M \) by \( m \rho m' \Leftrightarrow m \rho' m' \) for all \( \rho' \in W \). Then \( \rho \) is an S-congruence relation on \( M \).

**PROOF:**
(i) Reflexive:
Clearly \( m \rho m \) for all \( m \in M \) and for all \( \rho' \in W \)
\[ \Rightarrow m \rho m \text{ for all } m \in M \]

(ii) Symmetric
Let \( m \rho m' \)
\[ \Rightarrow m \rho' m' \text{ for all } \rho' \in W \]
\[ \Rightarrow m' \rho' m \text{ for all } \rho' \in W \Rightarrow m' \rho m \]

(iii) Transitive

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Let \( m \rho m' \) and \( m' \rho m'' \)
\[ \Rightarrow m \rho' m' \text{ and } m' \rho' m'' \text{ for all } \rho' \in W \]
\[ \Rightarrow m \rho' m' \text{ for all } \rho' \in W \Rightarrow m \rho m'' \]
Therefore \( \rho \) is an equivalence relation on \( M \).

(iv) Let \( m \rho m' \) and \( n \rho n' \)
\[ \Rightarrow m \rho' m' \land n \rho' n' \text{ for all } \rho' \in W \]
\[ \Rightarrow (m + n) \rho (m' + n') \land (s_1 s_2 m) \rho' (s_1 s_2 m') \text{ for all } \rho' \in W \text{ and for all } s_1 s_2 \in S \]
Hence \( \rho \) is a congruence relation on \( M \).

**NOTE 2.6.** \( \rho'' \leq \rho' \) for all \( \rho' \in W \) \( \Leftrightarrow \rho'' \leq \rho \)

**PROOF:** Suppose \( \rho'' \leq \rho' \) for all \( \rho' \in W \) \( \Leftrightarrow m \rho'' m' \Rightarrow m \rho' m' \text{ for all } \rho' \in W \)
\[ \Leftrightarrow m \rho'' m' \Rightarrow m \rho m' \Leftrightarrow \rho'' \leq \rho \]

**NOTE 2.7.** \((S - \text{Cong}(M), \leq)\) is a complete lattice.

**PROOF:** Clearly \((S - \text{Cong}(M), \leq)\) is a partially ordered set
Let \( \rho_1, \rho_2 \in S - \text{Cong}(M) \) then \( \rho_1 \cap \rho_2 \in S - \text{Cong}(M) \) and clearly \( \rho_1 \cap \rho_2 \leq \rho_1 \) and \( \rho_1 \cap \rho_2 \leq \rho_2 \) \( \Rightarrow \rho_1 \cap \rho_2 \) is the infimum of \( \{ \rho_1, \rho_2 \} \)
Similarly supremum of \( \{ \rho_1, \rho_2 \} \) is the supremum of \( \{ \rho_1, \rho_2 \} \)
Therefore \((S - \text{Cong}(M), \leq)\) is a lattice and for a family of S-congruences \( \{ \rho_i \}_{i \in I} \),
\[ \inf \{ \rho_i \} = \bigcap_{i \in I} \rho_i \text{ and } \sup \{ \rho_i \} = \bigcup_{i \in I} \rho_i \]
Hence \((S - \text{Cong}(M), \leq)\) is a complete lattice.

**NOTE 2.8.** Let \( m, m' \in M \), we denote the unique smallest element \( \rho \) of S-cong(M) satisfying
\[ m \rho m' \text{ by } \rho_{(m, m')} \]

**DEFINITION 2.11.** Let \( \rho \) be on S-congruence relation on \( M \) and \( M / \rho = \{ m / \rho | m \in M \} \). Define operations of addition and scalar multiplication by \( (m / \rho) + (n / \rho) = (m + n) / \rho \) and \( s_1 s_2 (m / \rho) = (s_1 s_2 m) / \rho \) for all \( m, n \in M \) and \( s_1 s_2 \in S \). Then \( M / \rho \) is a left ternary S-semimodule, called the factor ternary semimodule of \( M \) by \( \rho \).
Moreover, we have a surjective ternary S-homomorphism \( \Upsilon: M \rightarrow M / \rho \) defined by \( \Upsilon (m) = m / \rho \) with the operations \( \Upsilon (m_1 + m_2) = (m_1 + m_2) / \rho \) and \( \Upsilon (s_1 s_2 m) = (s_1 s_2 m) / \rho \)

**RESULT 2.1.** If \( M / \rho = M / \rho' \) if and only if \( \rho \text{ and } \rho' \) are equal.

**PROOF:** If \( \rho \) and \( \rho' \) are equal then \( M / \rho = M / \rho' \)
Suppose \( M / \rho = M / \rho' \)
Let \( (r, r') \in \rho \Rightarrow r \in r' / \rho \text{ and } r' \in r / \rho \)
Since \( M / \rho = M / \rho' \), there exists \( m \in M \) such that \( r' / \rho = r / \rho = m / \rho' \)
Thus \( r \in M / \rho' \) and \( r' \in M / \rho' \Rightarrow (r, m), (r', m) \in \rho' \Rightarrow (r, r') \in \rho' \)
\[ \Rightarrow \rho \subseteq \rho' \]
Similarly \( \rho' \subseteq \rho \)
Therefore \( \rho = \rho' \)
NOTE 2.9. If $N$ is a ternary subsemimodule of a left ternary $S$-semimodule $M$ and if $\rho \in S\text{-}\text{cong}(M)$, then the restriction of $\rho$ to $N$ is an $S$-congruence relation on $N$. Thus we have a canonical mapping $\gamma: S\text{-}\text{cong}(M) \to S\text{-}\text{cong}(N)$ given by restriction.

NOTE 2.10. If $\rho$ is an $S$-congruence relation on $M$, the restriction of which to $N$ is $\rho'$, then there is a monic $S$-homomorphism $\gamma: N/\rho' \to M/\rho$ defined by $\gamma(n/\rho') = m/\rho$.

PROOF: Define $\gamma: N/\rho' \to M/\rho$ by $\gamma(n/\rho') = m/\rho$.

To show $\gamma$ is well defined:

Let $n/\rho' = n'/\rho' \Rightarrow n\rho' = n'\rho' \Rightarrow n = n'$ such that

By reversing the above statements we get that $\gamma$ is one one.

To show $\gamma$ is $S$-homomorphism:

i) Consider $\gamma((n/\rho' + n'/\rho')) = \gamma(n + n'/\rho') = (n + n')/\rho = n/\rho + n'/\rho = \gamma(n/\rho') + \gamma(n'/\rho')$

ii) Consider $\gamma((s_1s_2.n/\rho')) = \gamma((s_1s_2.n)/\rho' = (s_1s_2.n)/\rho = s_1s_2(n/\rho) = s_1s_2\gamma(n/\rho')$

NOTE 2.11. In particular if the restriction of $\rho$ to $N$ that is $\rho'$ is trivial then the function $N \to M/\rho$ given by $n \mapsto n/\rho'$ is monic (since $N/\rho' = N$).

NOTE 2.12. If $\theta$ is an $S$-congruence relation on $M/\rho$ then $\theta$ defines a relation $\theta^*$ on $M$ by $m\theta^*m'$ if and only if $(m/\rho)\theta(m'/\rho)$. Then $\theta^*$ becomes an $S$-congruence relation on $M$ satisfying $\theta^* \geq \rho$. Moreover the function $\theta \to \theta^*$ is a morphism of complete lattices from $S\text{-}\text{cong}(M/\rho)$ to $S\text{-}\text{cong}(M)$.

PROOF: Define $\theta^*$ on $M$ by $m\theta^*m' \iff (m/\rho)\theta(m'/\rho)$

(i) Reflexive: Clearly $(m/\rho)\theta(m'/\rho)$ for all $m/\rho \in M/\rho \Rightarrow m\theta^*m$ for all $m \in M$

(ii) Symmetric: Suppose $m\theta^*m' \Rightarrow (m/\rho)\theta(m'/\rho) \Rightarrow (m'/\rho)\theta(m/\rho) \Rightarrow m'\theta^*m$ for all $m, m' \in M$

(iii) Transitive: Suppose $m\theta^*m'$ and $m'\theta^*m'' \Rightarrow (m/\rho)\theta(m'/\rho)$ and

$$(m'/\rho)\theta(m''/\rho) \Rightarrow (m/\rho)\theta(m''/\rho) \Rightarrow m\theta^*m''$$

for all $m, m', m'' \in M$

(iv) Suppose $m\theta^*m'; n\theta^*n'$

$$(m/\rho)\theta(m'/\rho)$$

and $(n/\rho)\theta(n'/\rho)$

and $[(m/\rho) + (n/\rho)]\theta[(m'/\rho) + (n'/\rho)]$ and

$s_1s_2(m/\rho)\theta s_1s_2(m'/\rho)$ and $s_1s_2(m/\rho)\theta s_1s_2(m'/\rho)$

$$(m + n)/\rho\theta(m + n')$$

and $(s_1s_2m/\rho)\theta(s_1s_2m'/\rho)$

Therefore $\theta^*$ is an $S$-congruence relation on $M$.

To prove $\theta^* \geq \rho$: Suppose $m\theta m' \Rightarrow m/\rho = m'/\rho$. 

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Since $\theta$ is reflexive, $(m/\rho)\theta (m'/\rho) \Rightarrow m\theta^*m'$
Therefore $m\rho m' \Rightarrow m\theta^*m'$. Hence $\rho \leq \theta^*$.

Now to prove the function $\theta \rightarrow \theta^*$ is a morphism of complete lattices from $\text{S-cong}(M/\rho)$ to $\text{S-cong}(M)$
Let $f : \text{S-cong}(M/\rho) \rightarrow \text{S-cong}(M)$ defined by $f(\theta) = \theta^*$
Clearly $f$ is well defined
To prove $f$ is a morphism of complete lattices:
(i) We have $f(\theta \land \theta'^*) = (\theta \land \theta'^*)^*$
Consider $m(\theta \land \theta'^*)^*m' = (m/\rho)(\theta \land \theta'^*)(m'/\rho)$
$= (m/\rho)(\theta^*(m'/\rho)) \land (m/\rho)((\theta^*)^*(m'/\rho))$
$= m\theta^*m' \land m\theta'^*m'$
$\Rightarrow m(\theta^* \land \theta'^*)m'$
Therefore $(\theta^* \land \theta'^*)^* = \theta^* \land \theta'^*$
$\Rightarrow f(\theta \land \theta'^*) = f(\theta) \land f(\theta'^*)$

(ii) We have $f(\theta \lor \theta'^*) = (\theta \lor \theta'^*)^*$
Consider $m(\theta \lor \theta'^*)^*m' = (m/\rho)(\theta \lor \theta'^*)(m'/\rho)$
$\Rightarrow \exists s_0 = m/\rho, s_1 s_2 s_n = m'/\rho$ and $\theta_1 \theta_2 \ldots \theta_n \in \{\theta; \theta'^*\}$ such that
$s_{i-1}s_i$ for all $1 \leq i \leq n$
$\Rightarrow \exists s_1 s_2 \ldots s_n \in \{\theta^*; \theta'^*\}$ such that
$t_{i-1} t_i$ for all $1 \leq i \leq n$
$\Rightarrow (m/\rho)(\theta \lor \theta'^*)(m'/\rho) = m(\theta^* \lor \theta'^*)m'$
$= m(\theta^* \lor \theta'^*)m'$ for all $m, m' \in M$
$\Rightarrow (\theta \lor \theta'^*)^* = \theta^* \lor \theta'^*$
$\Rightarrow f(\theta \lor \theta'^*) = f(\theta) \lor f(\theta'^*)$.
Hence the proof.

**DEFINITION 2.12.** A non zero ternary subsemimodule $N$ of a left ternary $S$-semimodule $M$ is absorbing if it satisfies the following conditions.
(i) If $0 \neq n \in N$ and $m \in M$ then $0 \neq n \oplus m \in N$
(ii) If $0 \neq n \in N$ then $(0:n) = \{0\}$

**RESULT 2.2.** Any non zero left ternary $S$-semimodule which is an absorbing semimodule itself is zero sum free.

**PROOF:** Let $M \neq \{0\}$ be absorbing left ternary $S$-semimodule.
To prove $M$ is zero sum free
Let $m + m' = 0$ and to show $m = m' = 0$
Suppose $m \neq 0$ that is $0 \neq m \in M$ and $m' \in M$ and $M$ is absorbing $\Rightarrow 0 \neq m + m' \in M$. a contradiction to $m + m' = 0$. So $m = 0$, Similarly $m' = 0$

**DEFINITION 2.13.** An element $m$ of a left ternary $S$-semimodule $M$ is cancellable if $m + m' = m + m'' \Rightarrow m' = m''$ and a ternary semimodule $M$ is said to be cancellable if every element of $M$ is cancellable.

**RESULT 2.3.** Any ternary $S$-module is cancellative.

**PROOF:** Let $M'$ be a ternary $S$-module and let $m \in M$ such that $m + m' = m + m''$
Therefore $M$ is a cancellative ternary semimodule.

**DEFINITION 2.14.** Let $M$ be a left ternary $S$-semimodule we put $(0; M) = \{x \in S/sxm = 0 \text{ for all } m \in M \text{ and for all } s \in S\}$

Then we call $(0; M)$ the annihilator of $M$ in $S$ denoted by $A_s(M)$.

**DEFINITION 2.15.** The zeroid of a ternary semiring $S$ denoted by $Z(S)$, is defined as

$Z(S) = \{s \in S \mid s = 0\}$

Clearly the zero element $0_s$ of $S$ is a member of $Z(S)$.

**LEMMA 2.1.** The zeroid $Z(S)$ of a ternary semiring $S$ is an h-ideal of $S$.

**PROOF:** Clearly $Z(S)$ is an ideal of $S$.

Suppose $x+y+z = y_1+z_1 = z_2$ where $x, z \in S$ and $y_1, y_2 \in Z(S)$

Now $x+y_1+z = y_2+z_2$ implies that $x+y_1+z_1+z = y_2+z_2+z_1+z$ which implies

$x+z_1+z+z_2 = z_2+z_1+z = z_1+z_2$.

This shows that $x \in Z(S)$. Hence $Z(S)$ is an h-ideal of $S$.

**LEMMA 2.2.** $A_s(M)$ is an h-ideal of a ternary semiring $S$.

**PROOF:** Clearly, $A_s(M)$ is an additive subsemigroup of $S$. Suppose $A_s(M)$ and for all $s$ and for all $t$.

Thus $s+st \in A_s(M)$ for all $s, t \in S$. Similarly we can show that $stx \in A_s(M)$.

**NOTE 2.13.** Let $M$ be a left ternary $S$-semimodule and $N$ be a ternary subsemimodule of $M$ then $N/m = \{x \in S/sxm \in N \text{ and for all } s \in S\}$

Moreover $N/m = \{x \in S/sxm \in N \text{ and for all } s \in S\}$

**PROPOSITION 2.1.** A necessary and sufficient condition to exist ternary $S$-semimodule which is absorbing ternary subsemimodule of itself is $S$ is entire and zero sum free.

**PROOF:** Suppose that $S$ is entire and zero sum free.

We know that $S$ is an ternary $S$-semimodule and $S$ is ternary subsemimodule of itself

Now we prove that $S$ is absorbing.

(i) Let $0 \neq s \in S$ and $s' \in S$ then $0 \neq s + s' \in S$

Otherwise if $s + s' = 0$ then $s = s' = 0$, contradiction to $S$ is zero sum free.

Therefore $0 \neq s + s' \in R$.

(ii) Let $0 \neq s \in S$ and $s \in (0:s)$

\[ ss'' = 0 \text{ for all } s'' \in S. \]

Since $S$ is entire $s = s' = s'' = 0 \Rightarrow s' = 0$

Therefore $(0:s) = \{0\}$ Hence $S$ is absorbing subsemimodule of itself.

Conversely suppose that $M$ be a ternary $S$-semimodule such that $M \subseteq M$

To prove $S$ is entire and zero sum free. Let $s, s' \in S$ such that $s + s' = 0$ and to prove $s = s' = 0$

Suppose $s \neq 0$ and let $m \neq 0$ then $ss'm \neq 0$ for $s' \in S$.

( if $ss'm = 0$ then $s \in (0;m) = \{0\} \Rightarrow s = 0$, a contradiction)
and \( s' sm \in M \Rightarrow 0 \neq s' m + s' sm \in M \)

Suppose \( s \neq 0 \) and Let \( m \neq 0 \) then \( s' sm \neq 0 \) for all \( s' \in S \),
If \( s' sm = 0 \) then \( s' \in (0; m) = \{0\} \)

**Definition 2.14.** An \( S \)-homomorphism \( \alpha : M \rightarrow N \) of ternary left \( S \)-semimodules is said to be steady if and only if the relations \( \equiv_\alpha \) and \( \equiv_{\ker \alpha} \) are coincide.

**Proposition 2.2.** Let \( S \) be a ternary semiring and \( N' \subseteq N \) be a ternary subsemimodule of a left ternary semimodule \( M \) then the function \( \alpha : M/N' \rightarrow M/N \) defined by \( \alpha(m/N') = m/N \) for all \( m \in M \)

To prove \( \alpha \) is steady surjective \( S \)-homomorphism

**Proof:** Suppose \( S \) is a ternary semiring and \( N' \subseteq N \) is ternary subsemimodules of a left ternary \( S \)-semimodule and \( \alpha : M/N' \rightarrow M/N \) defined by \( \alpha(m/N') = m/N \) for all \( m \in M \)

To prove \( \alpha \) is steady surjective \( S \)-homomorphism

(i) Consider \( \alpha((m + m')/N') = \alpha((m + m')/N') \)

\[
= (m + m') / N'
\]

\[
= m/N + m'/N
\]

\[
= \alpha(m/N') + \alpha(m'/N')
\]

(ii) Consider \( \alpha(r_1 r_2 m / N') = \alpha((r_1 r_2 m)/N') \)

\[
= (r_1 r_2 m) / N
\]

\[
= r_1 r_2 \alpha(m / N')
\]

Therefore \( \alpha \) is an \( S \)-homomorphism.

To prove \( \alpha \) surjective

Let \( m/N \in M/N \Rightarrow m \in M \)

\[ \Rightarrow \] there exists \( m/N' \in M/N' \) such that \( \alpha(m/N') = m/N \)

Therefore \( \alpha \) is surjective.

To prove \( \alpha \) steady

Clearly \( m/N' \equiv_{\ker \alpha} m'/N' \)

\[ \Rightarrow m/N' \equiv_{\alpha} m'/N' \]

Let \( m/N' \equiv_{\alpha} m'/N' \)

\[ \Rightarrow \alpha(m/N') = \alpha(m'/N') \]

\[ \Rightarrow m/N = m'/N \]

\[ \Rightarrow (m + n)/N' = (m' + n')/N' \]

\[ \Rightarrow \] there exists \( n, n' \in N \) such that \( m + n = m' + n' \)

\[ \Rightarrow m/N + n/N' = m'/N' + n'/N' \]

Now consider \( \alpha(n/N') = n/N = 0/N (\because n \in N) \)

Similarly \( \alpha(n'/N') = n'/N = 0/N (\because n' \in N) \)

\[ \Rightarrow \] there exists \( n, n', n'/N' \in \ker \alpha \) such that \( m/n + n/N' = m'/n' + n'/N' \)

\[ \Rightarrow m/N' \equiv_{\ker \alpha} m'/n' \]
therefore \( \equiv_\alpha \) and \( \equiv_{ker\alpha} \) are coincide.

Hence \( \alpha \) is steady surjective S-homomorphism.

**EXAMPLE 2.3.** Let \( N \) be an absorbing ternary subsemimodule of a left ternary S-semimodule \( M \). Define a relation \( \sim_N \) on \( M \) by \( m \sim_N m' \iff m = m' \) or both \( m, m' \in N \). Then \( \sim_N \) is an \( S \)-congruence relation on \( M \).

**LEMMA 2.3.** If \( \alpha: M \rightarrow N \) is an \( S \)-homomorphism of left ternary S-semimodules and if \( m, m' \in M \) such that \( M \cong_{ker\alpha} m' \) then \( m \cong_\alpha m' \)

**PROOF:** Let \( m \cong_{ker\alpha} m' \implies \) There exists \( n, n' \in ker\alpha \) such that \( m + n = m' + n' \)
\[ \implies \alpha(m + n) = \alpha(m' + n') \]
\[ \implies \alpha(m) + \alpha(n) = \alpha(m') + \alpha(n') \]
\[ \implies \alpha(m) + 0_N = \alpha(m') + 0_N \]
\[ \implies \alpha(m) = \alpha(m') \]
\[ \implies m \cong_\alpha m' \]

**LEMMA 2.4.** A steady \( S \)-homomorphism is monic if and only if \( ker\alpha = \{0\} \)

**PROOF:** Suppose \( \alpha: M \rightarrow N \), a steady \( S \)-homomorphism is monic

To prove \( Ker\alpha = \{0\} \).

Let \( m \in ker\alpha \implies \alpha(m) = 0_N = \alpha(0_M) \implies m = 0_M \implies ker\alpha = \{0\} \)

Conversely suppose that \( ker\alpha = \{0\} \)

To prove \( \alpha \) is monic : Let \( m, m' \in M \) such that \( \alpha(m) = \alpha(m') \)
\[ \implies m \cong_\alpha m' \]
\[ \implies m \cong_{ker\alpha} m' \]
\[ \implies m \cong_\alpha m' \]
\[ \implies m + 0 = m' + 0 \]
\[ \implies m = m' \]

Therefore \( \alpha \) is monic.

**DEFINITION 2.16.** Let \( \alpha: M \rightarrow N \) be an \( S \)-homomorphism of left ternary S-semimodules. Then we define the co-image of \( \alpha \) to be \( M/ker\alpha \) and the co kernel of \( \alpha \) to be \( N/(ker\alpha) \).

**PROPOSITION 2.3.** If \( M \) is a simple left ternary \( S \)-semimodule then \( M \) has no subtractive ternary submodules other than \( \{0_M\} \) and itself. The converse is true if \( M \) is left ternary \( S \)-module.

**PROOF:** Suppose \( M \) is a left ternary \( S \)-semimodule. Let \( N \) be a subtractive ternary subsemimodule of \( M \) and we have an \( S \)-congruence relation \( \equiv_N \) on \( M \). Since \( M \) is simple, \( \equiv_N \) is either trivial or universal.

Case (i) If \( \equiv_N \) is trivial then
\[ for \ n \in N, n + 0 = 0 + n \]
\[ \implies n \equiv_N 0_M \text{ and } \equiv_N \text{ is trivial, } n = 0_M \]
\[ \implies N = \{0_M\} \]

Case (ii) If \( \equiv_N \) is universal

Clearly \( N \subseteq M \)

Let \( m \in M \) and \( \equiv_N \) is universal
\[ \implies m \equiv_N 0_M \]
\[ \implies there exist n, n' \in N \text{ such that } m + n = 0_M + n' = n' \in N \]
Therefore M has no subtractive ternary subsemimodules other than \( \{0_M\} \) and itself.

Conversely suppose that M is a left ternary S-module satisfying the condition that \( \{0_M\} \) and itself are the only subtractive ternary submodules of M.

To prove M is simple. Let \( \rho \) be an S-congruence relation on M

Let \( N = \{m \in M \mid m \rho 0_M\} \)

Clearly \( 0_M \rho 0_M \Rightarrow 0_M \in M \Rightarrow N \neq \emptyset \)

Let \( m_1, m_2 \in N \) \( \Rightarrow m_1 \rho 0_M \) and \( m_2 \rho 0_M \) \( \Rightarrow (m_1 + m_2) \rho 0_M \Rightarrow m_1 + m_2 \in N \)

Let \( s_1, s_2 \in S \) and \( m \in N \) \( \Rightarrow m \rho 0_M \Rightarrow (s_1 s_2 m) \rho 0_M \Rightarrow s_1 s_2 m \in N \)

Therefore \( N \) is a ternary subsemimodule of M

Let \( m_1, m_1 + m_2 \in N \)

\[ m_1 \rho 0_M \text{ and } (m_1 + m_2) \rho 0_M \]

\[ m_1 / \rho = 0 / \rho \text{ and } (m_1 + m_2) / \rho = 0 / \rho \]

\[ m_1 / \rho = 0 / \rho \text{ and } m_1 / \rho + m_2 / \rho = 0 / \rho \Rightarrow 0 / \rho + m_2 / \rho = 0 / \rho \]

\[ m_2 / \rho = 0 / \rho \]

\[ m_2 \rho 0_M \Rightarrow m_2 \in N \]

Therefore \( N \) is subtractive.

By hypothesis \( N = \{0_M\} \) or \( N = M \)

Case(i) If \( N = \{0_M\} \) then \( 0_M \rho 0_M \)

Suppose \( m \rho m' \), for \( m \), there exist \( m \in M \) such that \( m + (-m) = 0_M \)

\[ m \rho m' \& m \not\rho m \]

\[ 0 \rho (m' - m) \]

\[ m' - m \in N = \{0_M\} \Rightarrow m' - m \]

Therefore \( \rho \) is trivial.

Case(ii) Suppose \( N = M \)

Then \( m \rho 0_M \) for all \( m \in M \) \( \Rightarrow m \rho 0_M \) and \( 0_M \rho m' \) for all \( m \), \( m' \in M \)

\[ m \rho m' \text{ for all } m, m' \in M \]

Therefore \( \rho \) is universal.

**PROPOSITION 2.4.** If \( N \) is a ternary subsemimodule of a ternary left S-semimodule M then the S-congruence relations \( \equiv_N \) and \( \equiv_{0/N} \) on M coincide.

**PROOF:** Given that \( N \) is a ternary subsemimodule of M

To prove \( \equiv_N \) and \( \equiv_{0/N} \) are coincide.

Let \( m \equiv_N m' \Rightarrow \) there exist \( n, n' \in N \) such that \( m + n = m' + n' \)

\[ m + n = m' + n' \Rightarrow \] there exist \( n, n' \in 0/N \) such that \( m + n = m' + n' (\because N \subseteq 0/N) \)

\[ m \equiv_{0/N} m' \]

Suppose \( m \equiv_{0/N} m' \Rightarrow \) there exist \( n, n' \in 0/N \) such that \( m + n = m' + n' \)

Now \( n, n' \in 0/N \Rightarrow \) there exist \( n_1, n_2, n_3, n_4 \in N \) such that \( n + n_1 = n_2 \) , \( n' + n_3 = n_4 \)

Now \( n + n_1 = n_2 \)

\[ \Rightarrow m + n + n_1 + n_3 = m' + n' + n_1 + n_3 \]

\[ \Rightarrow \text{there exist } n_2, n_3 \in N, n_1 + n_4 \in N \text{ such that } m + (n_2 + n_3) = m' + (n_1 + n_4) \]

\[ m + (n_2 + n_3) = m' + (n_1 + n_4) \]

Therefore \( \equiv_N m' \) and \( \equiv_{0/N} \) are coincide.
PROPOSITION 2.5. Let $S$ be a ternary semiring and let $M$ be a ternary left $S$-semimodule. Then a subset $N$ of $M$ is a subtractive ternary subsemimodule if and only if there exists an $S$-homomorphism $\alpha: m \rightarrow m'$ satisfying $N = \text{Ker } \alpha$

PROOF: Let $S$ be a ternary semiring and $M$ be a ternary left $S$-semimodule

Let $N \subseteq M$

Suppose that $N$ is a subtractive ternary subsemimodule of $M$

To prove there exists an $S$-homomorphism $\alpha: m \rightarrow m'$ such that $N = \text{Ker } \alpha$

Since $N$ is a ternary subsemimodule of $M$, $M' = M/N$ is a factor semimodule and we know $\alpha: M \rightarrow M'/N$ is an $S$-homomorphism defined by $\alpha(m) = m/N$

Consider $\text{Ker } \alpha = \{m \in M/\alpha(m) = 0/N\}$

Let $\{m \in M/m/N = 0/N\}$

Let $\{m \in M/\text{there exists } n, n' \in N\text{ such that } m + n = 0 + n'\}$

Let $\{m \in M/m + n = n' \in N, n \in N\}$ and $N$ is a subtractive

Therefore $N = \text{Ker } \alpha$

Conversely suppose that there exists an $S$-homomorphism, $\alpha: M \rightarrow M'$ such that $N = \text{Ker } \alpha$

To prove $N$ is a subtractive ternary subsemimodule of $M$. Clearly $\text{Ker } \alpha$ is a subtractive ternary subsemimodule of $M$ and $N = \text{Ker } \alpha$

Therefore $N$ is a subtractive ternary subsemimodule of $M$.

PROPOSITION 2.6. Let $S$ be a ternary semiring and let $\alpha: M \rightarrow N$ be an $S$-homomorphism of ternary left $S$-semimodules. If $N'$ is a subtractive ternary subsemimodule of $N$ and if $\alpha(M') = N' \subseteq M$ then

(i) $M'$ is a subtractive ternary subsemimodule of $M$ containing $\text{Ker } \alpha$; and

(ii) $\alpha$ induces an $S$-homomorphism $\beta: M/M' \rightarrow N/N'$ having kernel $\{0\}$

PROOF: Let $S$ be a ternary semiring, $\alpha: M \rightarrow N$ is an $S$-homomorphism and $N'$ is a subtractive ternary subsemimodule of $N$ and suppose $\alpha(M') = N' \subseteq M$

To prove $M'$ is a subtractive ternary subsemimodule of $M$

Let $m'm'' \in M' \Rightarrow \alpha(m'), \alpha(m'') \in N' = \{m \in M/\alpha(m) \in N\}$

$\Rightarrow \alpha(m') + \alpha(m'') \in N'$

$\Rightarrow \alpha(m' + m'') \in N'$

$\Rightarrow m' + m'' \in M'$

Therefore $M'$ is a ternary subsemimodule of $M$.

To prove $\text{Ker } \alpha \subseteq M'$:

Let $m \in \text{Ker } \alpha \Rightarrow \alpha(m) = 0$ and $0 \in N' \Rightarrow \alpha(m) \in N' \subseteq M$
To prove $M'$ Subtractive:

Let $m', m'' \in M$ such that $m', m' + m'' \in M'$

$\Rightarrow \alpha(m'), \alpha(m' + m'') \in N'$

$\Rightarrow \alpha(m'), \alpha(m') + \alpha(m'') \in N'$ and $N'$ is subtractive

$\Rightarrow \alpha(m'') \in N'$

$\Rightarrow m'' \in M'$

Therefore $M'$ is subtractive and hence $\alpha(M') = N' \subseteq M$ is a subtractive ternary subsemimodule of $M$ containing $\text{Ker} \, \alpha$

(ii) Define $\beta: m/M' \rightarrow N/N'$ by $\beta(m/M') = \alpha(m)/N'$

To prove $\beta$ is an $S$-homomorphism with $\text{ker} \, \beta = \{0/M'\}$

To show $\beta$ is well defined:

Let $m_1/M', m_2/M'$ such that $m_1/M' = m_2/M'$

There exists $m'_1, m'_2 \in M'$ such that $m_1 + m'_1 = m_2 + m'_2$

$\Rightarrow \alpha(m'_1 + m'_2) = \alpha(m_1 + m_2)$

$\Rightarrow \alpha(m_1) + \alpha(m'_1) = \alpha(m_2) + \alpha(m'_2)$

$\Rightarrow \text{there exists } \alpha(m'_1), \alpha(m'_2) \in N'$

such that $\alpha(m_1) + \alpha(m'_1) = \alpha(m_2) + \alpha(m'_2) \Rightarrow \alpha(m_1)/N' = \alpha(m_2)/N'$

$\Rightarrow \beta(m_1/M') = \beta(m_2/M')$

Therefore $\beta$ is well defined

To show $\beta$ is an $S$-homomorphism:

(i) Consider $\beta(m_1/M' + m_2/M') = \beta(m_1 + m_2/M') = \alpha(m_1 + m_2)/N'$

$= \alpha(m_1) + \alpha(m_2))/N'$

$= \alpha(m_1)/N' + \alpha(m_2)/N'$

$= \beta(m_1/M') + \beta(m_2/M')$

(ii) Consider $\beta(s_1s_2 (m/M')) = \beta((s_1s_2m)/M')$

$\Rightarrow \alpha(s_1s_2m)/N'$

$\Rightarrow (s_1s_2 \alpha(m))/N'$

$\Rightarrow s_1s_2 \beta(m/M')$

Therefore $\beta$ is an $S$-homomorphism

To prove $\text{Ker} \, \beta = \{0/M'\}$

Let $m/M' \in M / M'$ such that $\beta(m/M') = 0/N' \Rightarrow \alpha(m)/N' = 0 / N'$

there exist $n, n' \in N'$ such that $\alpha(m) + n = 0 + n'$

$\Rightarrow \alpha(m) + n \in N'$, $n' \in N'$ and $N'$ is subtractive

$\Rightarrow \alpha(m) \in N'$ [since $m \in \alpha^{-1}N^{-1} = M'$]

$\Rightarrow m \in M' \Rightarrow m/M' = 0/M'$
LEMMA2.5. Let $S$ be a ternary semiring and let $\alpha: M \to N$ be a surjective $S$-homomorphism of ternary left $R$-semimodules then there exist an $S$-semi isomorphism $M/\ker \alpha \to N$

**PROOF:** Let $S$ be a ternary semiring and $\alpha: M \to N$ be a surjective $S$-homomorphism. To prove there exists an $S$-semi isomorphism $M/\ker \alpha \to N$

Clearly $N' = \{0\}$ is a subtractive ternary sub semimodule of $N$ then

$\alpha(M') = N' = \{m \in M | \alpha(m) \in N'\}$

$= \{m \in M | \alpha(m) = 0\} = \ker \alpha$

Therefore $M' = \ker \alpha$

Then by proposition (2.6) there exists an $S$-homomorphism $\beta: M/M' \to N/N'$

such that $\ker \beta = \{0/M'\}$

$\Rightarrow$ there exists an $S$-homomorphism $\beta: M/\ker \alpha \to N/(0/\ker \alpha)$

$\Rightarrow$ there exists an $S$-isomorphism $\beta: M/\ker \alpha \to N$

LEMMA2.6. If $S$ is a ternary semiring and if $N' \subseteq N$ are ternary sub semimodules of a left $S$-semimodule $M$, then $M/N$ is $S$-isomorphic to $(M/N')/(N/N')$

**PROOF:** Let $S$ be a ternary semiring and $N' \subseteq N$ be ternary sub semimodules of a left $S$-semimodule $M$.

To prove $M/N$ is $S$-isomorphic to $(M/N')/(N/N')$

Define $\alpha: M/N'/N/N' \to M/N$ by $\alpha(m/N')/(N/N') = m/N$

Let $m_1, m_2 \in M$ such that $(m_1/N')/(N/N') = (m_2/N')/(N/N')$

$\Rightarrow m_1/N' + n_1/N' = m_2/N' + n_2/N'$ for some $n_1, n_2 \in N$

$\Rightarrow (m_1 + n_1)/N' = (m_2 + n_2)/N'$ for some $n_1, n_2 \in N$

$\Rightarrow m_1 + n_1 = m_2 + n_2$ for some $n_1, n_2 \in N$

$\Rightarrow$ there exists $n_1, n_2 \in N$

such that $m_1 + (n_1 + n_2) = m_2 + (n_2 + n_2)$

$\Rightarrow m_1/N = m_2/N$

$\Rightarrow \alpha(m_1/N')/(N/N') = \alpha(m_2/N')/(N/N')$

Therefore $\alpha$ is well defined.

To show $\alpha$ is an $S$-homomorphism

(i) Consider $\alpha[(m_1/N')/(N/N') + (m_2/N')/(N/N')]$

$= \alpha[(m_1/N' + m_2/N')/(N/N')]$

$= \alpha[(m_1 + m_2)/N']$

$= \alpha[(m_1/N) + (m_2/N)]$

$= \alpha[(m_1/N)/(N/N')] + \alpha[(m_2/N)/(N/N')]$

(ii) Consider $\alpha[s_1s_2((m/N')/(N/N'))] = \alpha[s_1s_2(m/N')/N/N']$

$= \alpha[(s_1s_2m/N')/(N/N')]$

$= s_1s_2m/N = s_1s_2(m/N)$

$= s_1s_2[(m/N')/(N/N')]$
Therefore $\alpha$ is an S-homomorphism

To prove $\alpha$ monic:
Let $m_1, m_2 \in M$ such that $\alpha[(m_1/N)/\langle N/N' \rangle] = \alpha[(m_2/N)/\langle N/N' \rangle]$

$\Rightarrow m_1/N = m_2/N$

$\Rightarrow m_1 + n = m_2 + n'$ for some $n, n' \in N$

$\Rightarrow m_1 + n + 0 = m_2 + n' + 0$ for some $n, n' \in N, 0 \in N'$

$\Rightarrow (m_1 + n)/N' = (m_2 + n')/N'$ for some $n, n' \in N$

$\Rightarrow m_1/N' + n/N' = m_2/N' + n'/N'$ for some $n/N', n'/N' \in N/N'$

$\Rightarrow (m_1/N')(N/N') = (m_2/N')/(N/N')$

Therefore $\alpha$ is monic

To prove $\alpha$ surjective:
Let $m/N \in M/N \Rightarrow m \in M$

$\Rightarrow (m/N')(N/N') \in (M/N')(N/N')$

such that $\alpha[(m/N')(N/N')] = m/N$

therefore $\alpha$ surjective

Hence $\alpha$ is isomorphism

PROPOSITION 2.7. Let $S$ be a ternary semiring. If $N$ and $N'$ are ternary subsemimodules of a
ternary left $S$-semimodule $M$, then there exists a canonical surjective S-homomorphism

$\alpha: N'/(N \cap N') \rightarrow (N + N')/N$, which is an $S$-semi isomorphism, if $N$ is subtractive.

PROOF: Define $\alpha: N'/(N \cap N') \rightarrow (N + N')/N$ by $\alpha[(N'/(N \cap N')]/N$.

To prove $\alpha$ is a surjective S-homorphism

To show $\alpha$ is well defined:
Let $n_1/(N \cap N'), n_2/(N \cap N') \in N/(N \cap N')$

such that $\alpha[n_1/(N \cap N')] = \alpha[n_2/(N \cap N')]$

$\Rightarrow$ there exist $x, y \in N \cap N'$ such that

$n_1 + x = n_2 + y$ $\Rightarrow$ there exist $x, y \in N$ such that $n_1 + x = n_2 + y$

$\Rightarrow n_1/N = n_2/N$

$\Rightarrow \alpha[n_1/(N \cap N')] = \alpha[n_2/(N \cap N')]$

Therefore $\alpha$ is well defined.

(i) Consider

$\alpha[n_1'/(N \cap N') + n_2'/(N \cap N')] = \alpha[(n_1' + n_2')/(N \cap N')]$

$= (n_1' + n_2')/N$

$= n_1'/N + n_2'/N$

$\Rightarrow \alpha[n_1'/(N \cap N') + \alpha[n_2'/(N \cap N')]$

(ii) Consider $\alpha[s_1s_2(n_1'/N \cap N')] = \alpha[(s_1s_2n_1')/N \cap N']$

$= (s_1s_2n_1')/N$

$= s_1s_2(n_1'/N)$
Therefore \( \alpha \) is a S-homomorphism:

Let \( x/N \in (N + N')/N \) where \( x = n + n' \), \( n \in N \) and \( n' \in N' \)
\[
\Rightarrow (n + n')/N \in (N + N')/N
\Rightarrow n/N + n'/N \in (N + N')/N
\Rightarrow 0/N + n'/N \in (N + N')/N \quad \text{since} \ n \in N
\Rightarrow n'/N \in (N + N')/N
\Rightarrow \text{there exists} \ n' \in N' \text{such that} \ \alpha((N'/N \cap N')) = n'/N = x/N
\]
Therefore \( \alpha \) is a surjective S-homomorphism

Suppose \( N \) is subtractive
To prove \( \alpha \) is an S-semi isomorphism
It is enough to prove that \( \ker \alpha = \{0/N \cap N'\} \)
Let \( n'/N \cap N' \in \ker \alpha \Rightarrow \alpha((n'/N \cap N')) = 0/N \Rightarrow n'/N = 0/N \)
\[
\Rightarrow n' + n_1 = 0 + n_2 \quad \text{for some} \ n_1, n_2 \in N
\Rightarrow n' + n_1 = n_2 \in N \quad \text{and} \ n_1 \in N \ & N \text{is subtractive}
\Rightarrow n' \in N
\Rightarrow n'/N \cap N' = 0/(N \cap N')
\Rightarrow \ker \alpha = \{0/(N \cap N')\}
\]
Therefore \( \alpha \) is an S-semi isomorphism.

**DEFINITION 2.17.** An element \( m \) of a left ternary S-semimodule \( M \) is cancellable if and only if \( m + m' = m + m'' \) implies that \( m' = m'' \). A ternary semimodule \( M \) is said to be cancellative if and only if every element of \( M \) is cancellable.

**NOTE 2.15.** Any ternary S-module is cancellable
**PROOF:** Let \( M' \) be an S-module
Let \( m \in M \) such that \( m + m' = m + m'' \)
Since \( V(M) = M \), there exists \( m \in M \) such that \( m + (-m) = 0_M \)
\[
\Rightarrow -m + m + m' = -m + m + m''
\Rightarrow m' = m''
\]
Therefore \( M \) is cancellative ternary semimodule.

**PROPOSITION 2.8.** The ternary left S-semimodule \( M \) is cancellative if and only if the ternary subsemimodule \( D = \{(m,m)/m \in M\} \) of \( M \times M \) is subtractive.
**PROOF:** Suppose that \( M \) is a ternary left S-semimodule \( D = \{(m,m)/m \in M\} \)
Also suppose that \( M \) is cancellative
To prove \( D \) is subtractive
Let \( (m,m), (m,m) + (m',m'') \in D \)
\[
\Rightarrow (m,m) \in D \quad \text{and} \ (m + m', m + m'') \in D \Rightarrow m + m' = m + m''
\Rightarrow m' = m''
\Rightarrow (m', m'') \in D
\]
Therefore \( D \) is subtractive.
Conversely suppose that \( D \) is subtractive
To prove \( M \) is cancellative
Let \( m \in M \) such that \( m + m' = m + m'' \Rightarrow (m + m', m + m'') \in D \)
Therefore $M$ is cancellative.

PROPOSITION 2.9. If $N$ is a ternary subsemimodule of a left ternary $S$-semimodule $M$ such that $0[/\]N\neq M$ then the Iizuka factor ternary semimodule $M[/\]N$ is cancellative.

PROOF: Let $m[/\]N\in M[/\]N$ such that $m[/\]N + m'[/\]N = m[/\]N + m''[/\]N.

$\Rightarrow m + m'[/\]N = m + m''[/\]N\Rightarrow$ there exist $n, n' \in N, m \in M$ such that $(m + m') + n + m_1 = (m + m'') + n' + m_1$

$\Rightarrow m' + n + (m + m_1) = m'' + n' + (m + m_1)$

There exists $n_1 n' \in N, m + m_1 \in M$ such that $m' + (m + m_1) = m'' + n' + (m + m_1)$

Therefore $M[/\]N$ is cancellative.

PROPOSITION 2.10. If $N$ is a ternary subsemimodule of a commutative left ternary $S$-semimodule $M$ then both $N$ and $M/N$ are cancellative.

PROOF: Suppose that $N$ is a ternary subsemimodule of a commutative left ternary $S$-semimodule $M$.

Let $n \in N$ such that $n + n' = n + n'', n', n'' \in N$ and $N \subseteq M$

$\Rightarrow n' = n''$ [since $n \in N \subseteq M$ and $M$ is cancellative]

Therefore $N$ is cancellative.

Let $m, m', m'' \in M$ such that $m/N + m'/N = m/N + m''/N$

$\Rightarrow (m + m')/N = (m + m'')/N$

$\Rightarrow$ there exists $n, n' \in N$ such that $m + m' + n = m + m'' + n'$

$\Rightarrow m' + n = m'' + n'$ (since $m \in M$ and $M$ is cancellative)

Therefore $M/N$ is cancellative.

EXAMPLE 2.4. If $M$ is ternary left $S$-semimodule then the relation $\theta$ defined by $m \theta m' \iff$ there exists an element $m'' \in M$ such that $m + m'' = m' + m''$ is an $S$-congruence relation on $M$ so, $M/\theta$ is a factor ternary $S$-semimodule, also $M/\theta$ is cancellative.

CONCLUSION: This manuscript is a study on factor ternary semimodules over ternary semiring using congruence relations. We established the following results i) necessary and sufficient condition to exist ternary $S$-semimodule which is absorbing subsemimodule of itself $S$ is entire and zero sum free. ii) If $S$ is a ternary semiring and $N' \subseteq N$ be a ternary subsemimodule of a left ternary semimodule $M$, then the function $\alpha : M/N' \rightarrow M/N$ defined by $\alpha(m/N') = m/N$ is steady surjective $S$-homomorphism iii) If $N$ is a ternary subsemimodule of a ternary left $S$-semimodule $M$ then the $S$-congruence relations $\equiv_N$ and $\equiv_0/N$ on $M$ coincide iv) If $N$ and $N'$ are ternary subsemimodules of a ternary left $S$-semimodule $M$, then there exists a canonical surjective $S$-homomorphism $\alpha : N'/N \cap N' \rightarrow (N + N')/N$, which is an $S$-semi isomorphism , if $N$ is subtractive.

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**AUTHOR'S BIOGRAPHY**

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