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Securing Multiagent Systems Against a Sequence of Intruder Attacks

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Abstract— **In this paper, we discuss the issue of security in multiagent systems in the context of their underlying graph structure that models the interconnections among agents. In particular, we investigate the minimum number of guards required to counter an infinite sequence of intruder attacks with a given sensing and response range of an individual guard. We relate this problem of eternal security in graphs to the domination theory in graphs, providing tight bounds on the number of guards required along with schemes for securing a multiagent system over a graph.**

I. INTRODUCTION

Security and protection against malicious agents and external intrusions is often required for a reliable operation of multiagent systems. This demands not only the proper surveillance of the system, but also the efficient response strategies to counter attacks within a suitable time span, thus, motivating the study of search and secure problems in cooperative systems.

Problems related to search and secure scenarios, have been studied in the literature under several settings, focusing on different aspects of the topic. They include the number of guards required for monitoring all the agents, a problem that is related to the art gallery problem (e.g. [1]), distributed detection schemes for observing abnormalities within agents (e.g., [2]–[4]), cooperative minimum time surveillance algorithms (e.g. [5]), and cooperative tracking of moving intruders with fixed sensors and mobile robots (e.g., [7], [8]), to name a few.

In multiagent systems, the interconnection among agents is frequently modelled by a graph structure where vertices (or nodes) represent agents and edges abstract the cooperation or interconnection among these agents. For the cases, where agents compute some values via pre-defined strategies, it is shown in [6] that the network topology completely characterizes the resilience of linear iterative strategies to malicious behavior of nodes. A graph theoretic interpretation of search and secure problems is of particular interest for multiagent systems, where protection of these cooperative systems is associated to the concept of security in graphs in some sense (e.g., [9]).

The notion of *eternal security* in graphs, introduced in [15], and later studied in [10] and [18], addresses the problem of making all the nodes in a graph secure against an infinite sequence of intruder attacks by a certain minimum number of guards. An *intruder attack* on a node (or a vertex) refers to any malicious activity on that node. A *guard* can detect and respond to an intruder attack by moving from one node to another in its neighborhood, along the edges of a graph. A dominating set in a graph is a set of vertices such that all the vertices lie in the immediate neighborhood of that set. Thus, placing guards at these dominating vertices ensures that every vertex is secured against an intruder. However, the movement of a guard from a dominating vertex to another, may leave some vertices unmonitored. In other words, the set of vertices corresponding to the new position of guards may not be a dominating set, thus leaving some vertices unsecured. The idea behind eternal security is to deal with such situations and secure vertices against an arbitrary sequence of attacks. This requires that vertices corresponding to the new positions of guards [1] also constitute a dominating set in a graph. Such a security in a graph is referred to as the *eternal 1-security* [10], where 1 denotes that only one guard moves in response to an attack while others maintain their current positions[2]. The minimum number of guards needed to make a graph eternally 1-secure is known as the *eternal 1-security number*, $\sigma_1(G)$, of a graph. Fig. 1 illustrates this concept through an example.

In this paper, we generalize the eternal 1-security problem by extending the notion of neighborhood to k-neighborhood. By this we mean that a guard can detect and respond to an intruder that is at most k-distant from it in a graph. This generalization, which we term as the *eternal 1-security with k-neighborhood*, allows us to study a relationship between the number of guards required and the distance covered by each guard to counter intruder attacks. Also, we can analyse the minimum number of guards required to eternally secure a graph, with a given sensing and response range of an individual guard. This analysis is often needed for designing cost effective and secure network topologies for multiagent and cooperative systems. See Fig. 3 for the illustration of an eternal 1-security of a given graph with 2-neighborhood.

The paper is organized as follows: In Section II, we introduce the necessary terms and notations. In Section III, we formally state the problem of eternal 1-security with k-neighborhood. Section IV presents some results on $\sigma_1^{(k)}(G)$ in terms of the graph power, G^k. In Section V, domination theory in graphs is used to obtain further results on the number of guards required. In Section VI, this problem is investigated for some classes of graphs. Finally, we present conclusions in Section VII.

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[1] after one of them moves to some node in its immediate neighborhood in response of an attack at that node.

[2] Another version of eternal security, known as the *eternal m-security*, also exists where m guards move in response to an attack [10].

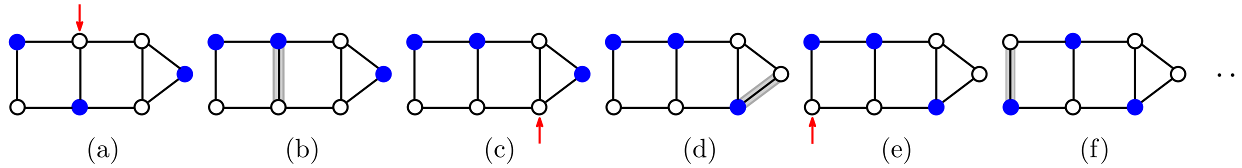

Fig. 1. (a) A given graph G is protected by 3 guards placed at the colored vertices. These vertices are in fact a dominating set of G. Let there be an intruder attack at the vertex, indicated by an arrow. (b) A guard in the neighborhood of an attacked vertex, moves towards it through a highlighted edge. Note that the set of vertices corresponding to the new guard positions, is still a dominating set of G. In (c), (d), (e) and (f), a sequence of intruder attacks and the response of the guards to counter them is shown. Note that the guards always lie on the vertices that dominate the whole graph, thus securing a graph against an infinite sequence of attacks.

II. PRELIMINARIES

A graph $G(V, E)$, with a vertex set $V(G)$ and an edge set $E(G)$, is a simple, undirected graph throughout this paper. A set $I \subset V(G)$ is an *independent set* of a graph if no two vertices in I are adjacent in G. The *independence number*, $\alpha(G)$, is the cardinality of a largest independent set. Let $W \in V(G)$ be a subset, such that every two vertices $x, y \in W$ are adjacent to each other in G (i.e. $(x, y) \in E(G)$), then the vertices in W induce a complete subgraph in G, referred to as a *clique*. The *clique cover number* of a graph, denoted by $\theta(G)$, is a partitioning of $V(G)$ into a minimum number of subsets such that the vertices in each subset induce a clique. The *distance* between two vertices $u, v \in V(G)$ in G, denoted by $d(u, v)_G$, is the length of the shortest path between u and v and the *diameter* of a graph, $diam(G)$, is max $d(u, v)_G, \ \forall u, v \in G$. The k^{th} *power of a graph* G, denoted by G^k, is a graph with $V(G^k) = V(G)$ and $(u, v) \in E(G^k)$, whenever $d(u, v)_G \leq k$.

The *open neighborhood* of a $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v, i.e., $N(v) = \{u \mid (u, v) \in E(G)\}$. The closed neighborhood of v, denoted by $N[v]$, is $N(v) \cup \{v\}$. Similarly, the open k-neighborhood of a vertex $v \in V(G)$, denoted by $N_k(v)$, is the set of vertices $\{v' \in V : d(v', v)_G \leq k\}$. The closed k-neighborhood, denoted by $N_k[v]$, is $N_k(v) \cup \{v\}$.

A set $S \in V(G)$ is a *dominating set*, if for each $v \in V(G)$, either $v \in S$, or v is adjacent to some $s_i \in S$. In other words, S is dominating if and only if $\bigcup_{s_i \in S} N[s_i] = V(G)$. The *domination number*, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set in G. For a connected graph, a *connected dominating set*, $S_{\gamma_c} \in V(G)$, is a dominating set such that the vertices in S_{γ_c}, induce a connected subgraph. The *connected domination number*, γ_c, is the cardinality of a minimum connected dominating set.

An example illustrating the above terms and notions is shown in the Fig. 2.

III. PROBLEM FORMULATION

Consider a graph where a certain number of guards are placed on its vertices. Every guard can detect and respond to an intruder attack on some vertex that is at most k-distant from it, by moving along the edges of a graph. We say that a vertex $v \in V$, is secured if there exists at least one guard that is at most k-distant from it. A graph is secured when all of its vertices are secured. In case of an attack on some vertex, a single guard will move to that vertex countering the attack. Now, if the graph remains secured with this new guard position along with the other guards that did not move, then all the vertices are secured against an infinite sequence of single vertex attacks, establishing the eternal 1-security of that graph with a k-neighborhood. An example illustrating the the eternal 1-security with 2-neighborhood in a given graph is shown in the Fig. 3. We can state the eternal 1-security with k-neighborhood formally as below.

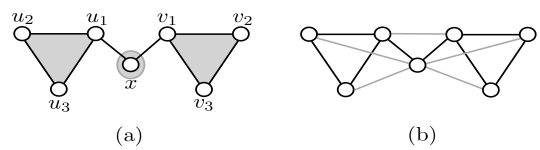

Fig. 2. (a) For a given G, an independent set $I = \{u_2, x, v_2\}$ with $\alpha(G) = 3$. Clique cover number, $\theta(G) = 3$, i.e., three cliques (highlighted in grey) span the whole graph. A dominating set $S = \{u_1, v_1\}$ with $\gamma(G) = 2$. A connected dominating set $S_{\gamma_c} = \{u_1, x, v_1\}$ with $\gamma_c(G) = 3$. (b) G^2, where extra edges due to the square of a graph are shown in grey.

Eternal 1-secure set with k-neighborhood of a graph G can be defined as a set $S_0 \in V(G)$ that can defend against any sequence of a single vertex attacks by a single guard shifts along the edges of G. It means that for any ℓ and any sequence of vertices $v_1, v_2, \cdots, v_\ell$, $\exists$ a sequence of guards $u_1, u_2, \cdots, u_\ell$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $d(u_i, v_i)_G \leq k$, such that each set $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is a dominating set with k-neighborhood. It should be noted that each S_i is an eternal 1-secure set with k-neighborhood, for all i. *Eternal 1-security number* of a graph G with *k-neighborhood*, denoted by $\sigma_1^{(k)}(G)$, is the cardinality of a smallest eternal 1-secure set with k-neighborhood.

In this paper, we analyse the $\sigma_1^{(k)}(G)$ for general graphs by giving tight bounds using various graph theoretic tools.

IV. $\sigma_1^{(k)}(G)$ AND THE GRAPH POWER, G^k

A fundamental lower and upper bounds for the eternal 1-security number of a graph with a usual notion of neighborhood (i.e., 1-neighborhood), were first presented in [15]. It relates $\sigma_1(G)$ with the independence number $\alpha(G)$ and the clique cover number $\theta(G)$, of a graph.

Theorem 4.1: [15] For any graph G,

$$\alpha(G) \leq \sigma_1(G) \leq \theta(G) \tag{1}$$

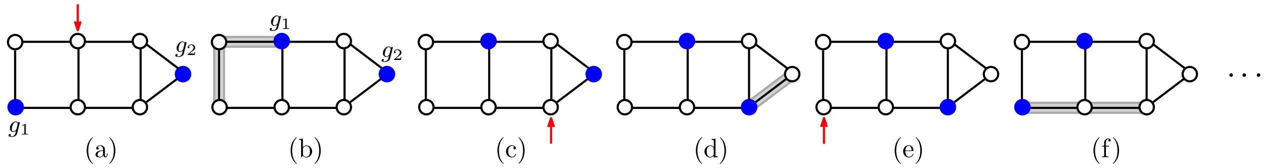

Fig. 3. (a) Two guards g_1 and g_2, are placed such that each $v \in V(G)$ is at most 2 edges away from at least one of the guards. Let there be an intruder attack at the vertex, indicated by an arrow. (b) The guard, g_1, counters the attack by moving through a highlighted path of two edges. (c) Note that even in a new position of guards, each $v \in V(G)$, is at most two edges away from at least one of the guards. In (c), (d), (e) and (f), a sequence of intruder attacks and the response of the guards to counter them is shown.

Due to their significance as one of the primary notions in graph theory, literature has many nice results regarding $\alpha(G)$ and $\theta(G)$. Thus, the above theorem provides a way of translating known results for the topic of eternal 1-security. In a similar way, it will be useful to interpret eternal 1-security with k-neighborhood, in terms of the usual notion of eternal 1-security, allowing us to use already known results for interpreting this generalized notion of eternal security in graphs. Theorem 4.2 relates $\sigma_1^{(k)}(G)$ and $\sigma_1(G)$.

Theorem 4.2: A graph G is eternally 1-secure with k-neighborhood, if and only if G^k is eternally 1-secure with 1-neighborhood, where G^k is the k^{th} power of a graph G.

Proof: Eternal 1-security of G with k-neighborhood impiles that, for any ℓ and any sequence of vertices $v_1, v_2, \cdots, v_\ell$ in G, $\exists$ a sequence of guards $u_1, u_2, \cdots, u_\ell$ such that either $u_i = v_i$ or $d(u_i, v_i)_G \leq k$. In G^k, $d(u_i, v_i)_{G^k} \leq 1$ whenever $d(u_i, v_i)_G \leq k$. Thus eternal 1-security of G with k-neighborhood implies the existence of a guard u_i for any vertex v_i in G^k, such that $d(u_i, v_i)_{G^k} \leq 1$, for any i and any sequence of vertices, implying the eternal 1-security of G^k with 1-neighborhood.

Since $(u_i, v_i) \in E(G^k)$ implies that $d(u_i, v_i)_G \leq k$. Thus, using the same argument as above, eternal 1-security of G with 1-neighborhood is directly implied from the eternal 1-security of G^k with 1-neighborhood. ■

Following lemma is a direct consequence of the Theorem 4.2.

Lemma 4.3: For any graph G,

$$\sigma_1^{(k)}(G) = \sigma_1(G^k)$$

Theorem 4.4: For any graph G and positive integers m and n,

$$\sigma_1^{(m)}(G^n) = \sigma_1^{(n)}(G^m) \tag{2}$$

Proof: Let $G^n = X$ and $G^m = Y$. Using Lemma 4.3, left hand side of (2) becomes, $\sigma_1^{(m)}(X) = \sigma_1(X^m) = \sigma_1(G^{nm})$, and right side gives, $\sigma_1^{(n)}(Y) = \sigma_1(Y^n) = \sigma_1(G^{nm})$. Thus, we get the required result. ■

A simple, but an important result from [13] states that $\sigma_1(G) = 1$, if and only if G is a complete graph. Using this fact, we obtain the following result.

Theorem 4.5: For a connected graph G, $\sigma_1^{(k)}(G) = 1$, if and only if $k \geq diam(G)$.

Proof: Let $\sigma_1^{(k)}(G) = 1$. From Lemma 4.3, $\sigma_1(G^k) = 1$. Also, from [13], we get that G^k is a complete graph, i.e. $(v_i, v_j) \in E(G^k)$, $\forall v_i, v_j \in V(G^k)$, where $i \neq j$. This implies $d(v_i, v_j)_G \leq k$ in G for all $v_i, v_j \in V(G)$, where $i \neq j$, which means that $diam(G) \leq k$.

Let $k \geq diam(G)$. Then G^k is a complete graph. From [13], σ_1 of a complete graph is 1. Thus, $\sigma_1(G^k) = \sigma_1^{(k)}(G) = 1$. ■

An immediate consequence of the Theorem 4.5 is the following,

Corollary 4.6: $\sigma_1^{(k)}(G) = 1$, if and only if G^k is a complete graph. ■

The above results also provide a systematic way of finding $\sigma_1^{(k)}(G)$, for an arbitrary graph. For a given k, consider a low diameter decomposition of a graph G, where each connected component has a diameter at most k. By this we mean a partitioning of $V(G)$ into subsets $V(G) = \bigcup_{i=1}^{a} V_i$, such that the subgraph induced by each subset V_i has a diameter at most k. Then, $\sigma_1^{(k)}(G)$ is at most equal to the number of components in the decomposition (or the number of subsets in the partitioning of $V(G)$). This is so because each component has a diameter at most k, and therefore, has an eternal 1-security number with k-neighborhood equal to 1 (using Theorem 4.5). Conversely, if it is desired to find a suitable k for a given $\sigma_1^{(k)}(G)$,[3] then we need to decompose a graph into $\sigma_1^{(k)}(G)$ number of connected components. Then, the diameter of the component with the maximum diameter will be the required k.

As an example, consider a case where a graph G is decomposed into k-caterpillars. A caterpillar is a tree where every vertex lies either on a central path, or at a distant 1 from some vertex on a central path. A k-caterpillar is a caterpillar with a central path of k vertices. An example is shown in the Fig. 4.

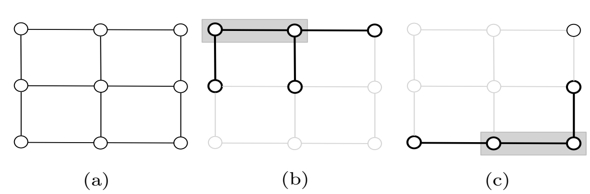

Fig. 4. A graph G is shown in (a). It can be decomposed into two 2-caterpillars as shown in (b) and (c). Central paths of two vertices in 2-caterpillars are highlighted in (b) and (c).

[3]This situation arises when a fixed number of guards are available and it is desired to find out the required sensing and response range of a guard.

It is to be noted that $diam(\mathcal{C}_\ell) = \ell + 1$, where $\mathcal{C}_\ell$ is an ℓ-caterpillar. Thus $\sigma_1^{(\ell+1)}(\mathcal{C}_\ell) = 1$, using Theorem 4.5. Now, if we get a spanning subgraph of a given graph G, such that each connected component in that spanning subgraph is an ℓ-caterpillar with $\ell \leq (k-1)$, then, $\sigma_1^{(k)}(G)$ will always be lesser than or equal to the number of components in that spanning subgraph. This is true as, $\sigma_1^{(k)}(\mathcal{C}_\ell) = 1$ for each connected component of the spanning subgraph (that is an ℓ-caterpillar with $\ell \leq k-1$). Thus, a caterpillar decomposition of a graph, that is a partitioning of $V(G)$ into subsets, where the vertices in each subset are spanned by a caterpillar, provides a method for finding a sufficient number of guards for the eternal 1-security of a graph with k-neighborhood.

V. ETERNAL 1-SECURITY AND DOMINATION IN GRAPHS

Domination in graphs has been extensively studied in the graph theory literature. Several variants of the domination concept exists, including distance domination, total domination and connected domination, to name a few. Since, a lot of theoretical and algorithmic results are available for various versions of domination (see [17] for details), thus, relating the notion of eternal 1-security in graphs to these domination related concepts turns out to be useful for the computation of $\sigma_1^{(k)}(G)$. In this section, we relate $\sigma_1^{(k)}(G)$ to the notions of k-distance domination, total k-distance domination and the connected domination in graphs.

We start by relating $\sigma_1^{(k)}(G)$ with the k-distance domination number, $\gamma^{(k)}(G)$ of a graph. A k-distance dominating set, or simply a *k-dominating set*, $S^{(k)} \in V(G)$, is a set of vertices such that for each $v \in V(G)$, either $v \in S^{(k)}$ or v is at most k distant from some vertex in $S^{(k)}$. The cardinality of a minimum k-dominating set is the *k-domination number* of a graph, denoted by $\gamma^{(k)}$.

Theorem 5.1: For any graph G,

$$\sigma_1^{(2k)}(G) \leq \gamma^{(k)}(G) \tag{3}$$

where $\gamma^{(k)}$ is a k-domination number of G.

Proof: Let $S^{(k)} = \{s_1, s_2, \cdots, s_{\gamma^{(k)}}\}$ be a minimum k-dominating set of G. Let G_{s_i} be a subgraph induced by the vertices in $N_k[s_i]$. Then, by the definition of a k-dominating set, $d(v, s_i)_{G_{s_i}} \leq k$, $\forall v \in N_k[s_i]$. Thus, for any $x, y \in N_k[s_i]$, $d(x, y)_{G_{s_i}} \leq 2k$, implying that $diam(G_{s_i}) \leq 2k$. By using Theorem 4.5, $\sigma_1^{(2k)}(G_{s_i}) = 1$. This is true for each $s_i \in S^{(k)}$. Since, $\bigcup_{s_i \in S^{(k)}} G_{s_i} \subseteq G$, so we get $\sigma_1^{2(k)}(G) \leq \gamma^{(k)}$. ■

For $k = 1$, we have $\sigma_1^{(2)}(G) \leq \gamma(G)$, where, $\gamma(G)$ is a domination number of a graph. An example illustrating the above proof for $k = 1$ is shown in the Fig. 5. It is to be mentioned here that the bound in (3) is tight. For example, consider the graph in the Fig. 5, where $\sigma_1^{(2)}(G) = \gamma(G) = 2$.

Using Theorem 5.1 and the notion of graph power, G^k, we can generalize the relationship between $\sigma_1^{(k)}(G)$ and the domination number of a graph.

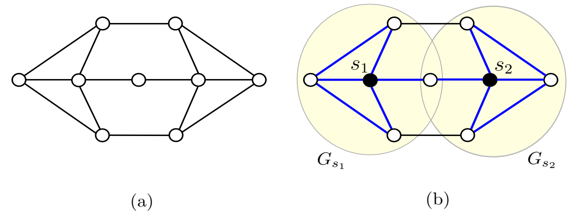

Fig. 5. (a) Given a graph G. (b) $S = \{s_1, s_2\}$ is a dominating set. For each $s_i \in S$, there exits a subgraph G_{s_i}, with $diam(G_{s_i}) = 2$, and so, $\sigma_1^{(2)}(G_{s_i}) = 1$. Also, $G_{s_1} \cup G_{s_2} \subseteq G$. So, $\sigma_1^{(2)}(G) \leq \left[\sigma_1^{(2)}(G_{s_1}) + \sigma_1^{(2)}(G_{s_2})\right] = 2$.

Theorem 5.2: For any graph G and a positive k,

$$\sigma_1^{(2k)}(G) \leq \gamma(G^k) \tag{4}$$

Proof: Let $G^k = Z$. Using Lemma 4.3 we get, $\sigma_1^{(2k)}(G) = \sigma_1(G^{2k}) = \sigma_1(Z^2) = \sigma_1^{(2)}(Z)$. Now using Theorem 5.1, $\sigma_1^{(2)}(Z) \leq \gamma(Z)$, i.e. $\sigma_1^{(2)}(G^k) \leq \gamma(G^k)$, implying $\sigma_1^{(2k)}(G) \leq \gamma(G^k)$. ■

We can also relate $\sigma_1^{(k)}(G)$ to a widely studied notion of connected domination in graphs.

Theorem 5.3: For any connected graph G,

$$\sigma_1^{(k+2)}(G) \leq \sigma_1^{(k)}(G_{\gamma_c})$$

where, G_{γ_c} is a subgraph induced by the vertices in a minimum connected dominating set, S_{γ_c}, of G.

Proof: Let $S_{\gamma_c} = \{s_1, s_2, \cdots, s_{\gamma_c}\}$ be a set of vertices in a minimum connected dominating set of G and G_{γ_c} be a subgraph induced by S_{γ_c}.

A vertex $v \in V(G)$, is eternally 1-secure with k-neighborhood, if there always exists a guard u, such that $d(u, v)_G \leq k$. Then, a graph is eternally 1-secure with k-neighborhood, if and only if all of its vertices are eternally 1-secure with k-neighborhood. Now, let us assume that G has $\sigma_1^{(k)}(G_{\gamma_c})$ number of guards.

Claim 1: Each $s_i \in S_{\gamma_c}$ is eternally 1-secure with $(k+1)$-neighborhood in G.

Proof: $\sigma_1^{(k)}(G_{\gamma_c})$ guards will ensure that, for each $s_i \in V(G_{\gamma_c})$, there always exists a guard in G_{γ_c} that will eternally 1-secure it with a k-neighborhood. Now in G, for every $v \in V - S_{\gamma_c}$, there always exist some $s \in S_{\gamma_c}$ such that $d(v, s)_G = 1$. This is true as S_{γ_c} is a dominating set of G. So, in case of an attack on some $s_i \in S_{\gamma_c}$, there always exist a guard in G that is at most $k+1$ distance away from it.

Claim 2: Each $v_i \in V(G) - S_{\gamma_c}$ is eternally 1-secure with $(k+2)$-neighborhood in G.

Proof: Let there be an attack at some $v_i \in V - S_{\gamma_c}$, and s_i be a vertex in S_{γ_c} such that $d(v_i, s_i)_G = 1$. By claim 1, for every $s_i \in S_{\gamma_c}$, there exists a guard in G that is at most $k+1$ distant from it. Thus, there always exists a guard in G that is at most $k+2$ distance away from $v_i \in V - S_{\gamma_c}$, making every such v_i eternally 1-secure with $(k+2)$-neighborhood in G. □

From claims 1 and 2, all the vertices in G are eternally 1-secure with $(k+2)$-neighborhood. ■

A. Eternal 1-Security and the Total k-Domination in Graphs:

A set $S_{\gamma_t^k}(G)$ is a *total k-dominating set*, if every $v \in V$ is within a distance k from some vertex of $S_{\gamma_t^k}(G)$, other than itself. The cardinality of a smallest set $S_{\gamma_t^k}(G)$, is known as the *total k-domination number*, $\gamma_t^k(G)$, of a graph.

Now, we will relate $\sigma_1^{(k)}(G)$ with a total k-domination number of a graph. First, we define the following notion.

Definition 5.1: (Matching in a Graph): Given a graph G, a matching M, is a set of edges that do not share a common vertex. The cardinality of a largest matching in a graph is called the *matching number* of G, denoted by $\nu(G)$. Also a vertex is *matched* if it is incident to an edge in a matching, otherwise a vertex is *unmatched*.

Theorem 5.4: For any connected graph G, and $k \geq 1$,

$$\sigma_1^{(2k+1)}(G) \leq \gamma_t^k(G) - \nu(G_{\gamma_t^k})$$

where, $\nu(G_{\gamma_t^k})$ is a matching number of a subgraph induced by the vertices in a minimum total k-dominating set, $S_{\gamma_t^k}(G)$.

Proof: Let $S_{\gamma_t^k} = \{s_1, s_2, \cdots, s_{\gamma_t^k}\}$ be a minimum total k-dominating set of a G. If G_{s_i} is a subgraph induced by the vertices in $N_k[s_i]$, then, by the definition of a total k-dominating set, $diam(G_{s_i}) \leq 2k$. Now, let $G_{\gamma_t^k}$ be a subgraph induced by the vertices in $S_{\gamma_t^k}$, and M be a maximum matching of $G_{\gamma_t^k}$ with $\mid M \mid = \nu(G_{\gamma_t^k})$. Then, without loss of generality, we can partition $S_{\gamma_t^k} = \{s_1, s_2\} \cup \{s_3, s_4\} \cup \cdots \{s_{(2\nu-1)}, s_{(2\nu)}\} \cup \{s_{(2\nu+1)}, \cdots, s_{\gamma_t^k}\}$, where the vertices in $\{s_i, s_{i+1}\}$ for $i+1 < (2\nu+1)$, are the end vertices of some edge $e \in M$, and the vertices in $\{s_{(2\nu+1)}, \cdots, s_{\gamma_t^k}\}$ are the unmatched vertices. Then, $G \supseteq \bigcup\limits_{s_i \in S_{\gamma_t^k}} G_{s_i}$

$$= (G_{s_1} \cup G_{s_2}) \cup (G_{s_3} \cup G_{s_4}) \cup \cdots (G_{s_{2\nu-1}} \cup G_{s_{2\nu}}) \bigcup_{i=2\nu+1}^{\gamma_t^k} G_{s_i}$$

Noting that $diam(G_{s_i} \cup G_{s_{i+1}}) \leq (2k+1)$, for $i+1 < 2\nu+1$. Thus, we decompose a given G into $\nu + (\gamma_t^k - 2\nu)$ components, where the diameter of each component is at most $(2k+1)$. Using Theorem 4.5, the $\sigma_1^{(2k+1)}$ of each component is 1. Thus, we get $\sigma_1^{(2k+1)}(G) \leq \nu + (\gamma_t^k - 2\nu) = \gamma_t^k - \nu(G_t^k)$, which is the desired result. ■

An illustration of the above proof, through an example is shown in the Fig. 6.

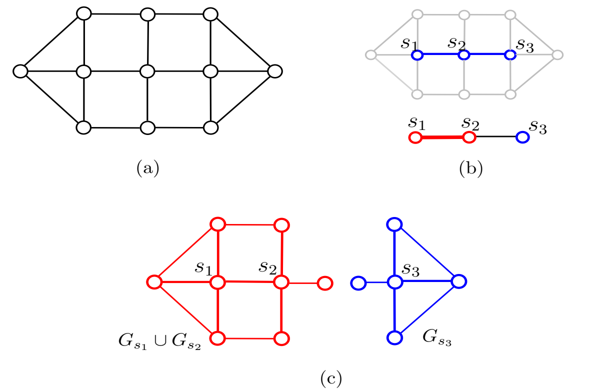

Fig. 6. (a) Given a graph G. (b) $S_{\gamma_t^1} = \{s_1, s_2, s_3\}$ is a total dominating set. $G_{\gamma_t^1}$ (shown in dark) is the subgraph induced by the vertices in $S_{\gamma_t^1}$. Matching of $G_{\gamma_t^1}$ along with the matched vertices, $\{s_1, s_2\}$ is also shown. (c) Note that $diam(G_{s_1} \cup G_{s_2}) = 3$ and $diam(G_{s_3}) = 2$, so $\sigma_1^{(3)}(G_{s_1} \cup G_{s_2}) = \sigma_1^{(3)}(G_{s_3}) = 1$. This gives, $\sigma_1^{(3)}(G) \leq 2$, which is same as $\gamma_t^1(G) - \nu(G_{\gamma_t^1}) = 3 - 1 = 2$.

B. Eternal 1-Security and the k-Distance Paired Domination:

A set $S_{\gamma_t^k}(G)$, is a *k-distance paired dominating set*, if it is a k-dominating set, and the subgraph induced by the vertices in $S_{\gamma_p^k}(G)$ has a perfect matching. The cardinality of a smallest $S_{\gamma_p^k}(G)$, is known as the *k-distance paired domination number*, $\gamma_p^k(G)$, of a graph (See [14] for details).

Theorem 5.5: If G is a connected graph, then,

$$\sigma_1^{(2k+1)}(G) \leq \frac{\gamma_p^k}{2}$$

Proof: Let $S_{\gamma_p^k} = \{s_1, s_2, \cdots, s_{\gamma_p^k}\}$ be a minimum k-distance paired dominating set of a given G, and $G_{\gamma_p^k}$ be a subgraph induced by the vertices in $S_{\gamma_p^k}$. Since, there exists a matching, M, of $G_{\gamma_p^k}$, such that all of its vertices are matched, so, we can do partition $S_{\gamma_p^k} = \{s_1, s_2\} \cup \{s_3, s_4\} \cdots, \cup \{s_{(\gamma_p^k-1)}, s_{\gamma_p^k}\}$, where s_i is connected to s_{i+1} in each subset $\{s_i, s_{i+1}\}$. Also, if G_{s_i} is a subgraph induced by the vertices in $N_k[s_i]$, then, by the definition of the k-distance paired dominating set, $diam(G_{s_i}) \leq 2k, \forall s_i \in S_{\gamma_p^k}$. Thus, $diam(G_{s_i} \cup G_{s_{i+1}}) \leq 2k+1$, where s_i and s_{i+1} are the end vertices of a same edge in M. Thus, $G \supseteq \bigcup\limits_{s_i \in S_{\gamma_p^k}} G_{s_i} = (G_{s_1} \cup G_{s_2}) \cup (G_{s_3} \cup G_{s_4}) \cdots \cup (G_{s_{(\gamma_p^k-1)}} \cup G_{\gamma_p^k})$. So, we can decompose G into $\frac{\gamma_p^k}{2}$ components where diameter of each component is at most $(2k+1)$, and thus, $\sigma_1^{(2k+1)}$ is 1 for each component. This gives, $\sigma_1^{(2k+1)}(G) \leq \sum\limits_{i=1}^{\gamma_p^k/2} 1 = \frac{\gamma_p^k}{2}$, which is the required result. ■

VI. Eternal 1-Security with k-Neighborhood for Some Classes of Graphs

In this section, we give expressions for $\sigma_1^{(k)}(G)$ for path and cycle graphs. We start with a path graph by first stating the notion of chromatic number, $\chi(G)$, that will be used.

Definition 6.1: (Chromatic Number, $\chi(G)$): The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices so that no two adjacent vertices share the same color.

Theorem 6.1: Let G be a path graph P_n having n vertices, then,

$$\sigma_1^{(k)}(P_n) = \left\lceil \frac{n}{k+1} \right\rceil$$

Proof: From [11], $\chi(P_n^k) = k+1$. Since, $\alpha \geq \frac{n}{\chi}$ [11], thus, $\alpha(P_n^k) \geq \frac{n}{k+1}$ and using (1), we imply that $\sigma_1(P_n^k) \geq \lceil \frac{n}{k+1} \rceil$. Also in P_n^k, every $(k+1)$ consecutive vertices make a complete subgraph. Thus, we get $\lceil \frac{n}{k+1} \rceil$ cliques implying

$\sigma_1(P_n^k) \leq \lceil \frac{n}{k+1} \rceil$. Now, observing that $\sigma_1(P_n^k) = \sigma_1^{(k)}(P_n)$, we get the desired result. ■

Eternal 1-security number with k-neighborhood of a circle graph is of particular interest, as a bounded region can always be enclosed by a circle graph.

Theorem 6.2: Let G be a cycle graph C_n with n vertices, then,

$$\left\lfloor \frac{n}{k+1} \right\rfloor \leq \sigma_1^{(k)}(C_n) \leq \left\lceil \frac{n}{k+1} \right\rceil$$

Proof: From [12], we know that $\alpha(C_n^k) = \lfloor \frac{n}{k+1} \rfloor$, thus using Theorem 4.1 and Lemma 4.3, we get $\sigma_1^{(k)}(C_n) \geq \lfloor \frac{n}{k+1} \rfloor$. Now assume that vertices of C_n are labelled consecutively $\{1, 2, \cdots, n\}$. Consider a partition $\mathcal{P}$, of $V(C_n^k)$.

$$\begin{aligned}\mathcal{P} =& \{\{1, \cdots, k+1\},\ \ \{k+2, k+3, \cdots, 2(k+1)\},\\ & \cdots, \{x, x+1, \cdots, n\}\}\end{aligned}$$

where, $x = \left[\left(\lceil \frac{n}{k+1} \rceil - 1\right)(k+1) + 1\right]$. Note that, all the vertices in each subset of $\mathcal{P}$ are adjacent to each other in C_n^k. Thus, the vertices in each subset of $\mathcal{P}$ induce a clique in C_n^k. Also, the cardinality of $\mathcal{P}$ is $\lceil \frac{n}{k+1} \rceil$. This gives a clique cover number of C_n^k, i.e. $\theta(C_n^k) = \lceil \frac{n}{k+1} \rceil$. $\sigma_1^{(k)}(C_n) \leq \lceil \frac{n}{k+1} \rceil$ is then directly implied by Theorem 4.1. ■

An example illustrating the above proof is shown in the Fig. 7.

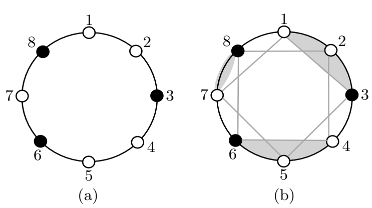

Fig. 7. (a) C_8 with vertices labelled consecutively $\{1, 2, \cdots, 8\}$. (b) In C_8^2, the vertices in each of the subsets, $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8\}$, induce a complete subgraph, highlighted in grey. Note that there is a guard for each complete subgraph in C_8^2.

Following result is a direct consequence of the Theorem 6.2.

Corollary 6.3: Every hamiltonian graph[4] has,

$$\sigma_1^{(k)}(G) \leq \left\lceil \frac{n}{k+1} \right\rceil$$

Another useful result directly follows from the Theorem 6.2 and Corollary 6.3.

Corollary 6.4: For $k \geq 2$, every 2-connected graph[5] has,

$$\sigma_1^{(k)}(G) \leq \left\lceil \frac{n}{k+1} \right\rceil$$

Proof: If G is 2-connected, then G^2 is hamiltonian [16]. Corollary 6.3 directly implies the required result. ■

[4] A hamiltonian cycle in a graph G, is a cycle that passes through each vertex exactly once. A graph containing such a cycle is a hamiltonian graph.

[5] A graph is 2-connected if there does not exist a single vertex whose removal disconnects the graph.

VII. CONCLUSIONS

In this work, we investigated the issue of security in multiagent systems from a graph theoretic view point. We proposed a framework, where a certain minimum number of guards secure a multiagent system against an infinite sequence of intruder attacks over a graph that models the underlying inter-connections among agents. Under this setup, we analysed the number of guards required for securing a graph structure, with a given sensing and response range of an individual guard. This also allowed us to relate the maximum distance a guard needs to move to counter an intruder attack when the number of guards is fixed. Moreover, an analysis performed for various classes of graphs is helpful for designing secure and reliable network topologies for multiagent systems.

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