Math 1497 - Calc 2

Taylor Polynomials

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$$

$\vdots$

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Ex 1 Find $P_3(x)$ for $f(x) = \sqrt{x}$ at $x = 4$

$f(x) = x^{1/2}$ $\qquad f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$ $\qquad f''(4) = -\frac{1}{4} \cdot \frac{1}{4^{3/2}} = -\frac{1}{32}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$ $\qquad f'''(4) = \frac{3}{8} \cdot \frac{1}{4^{5/2}} = \frac{3}{8} \cdot \frac{1}{32} = \frac{3}{256}$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\frac{(x-4)^2}{2!} + \frac{3}{256} \cdot \frac{(x-4)^3}{3!}$$

if we let $n \to \infty$

$$f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

this is called a Taylor Series

and we can test this power series

Ex 2 $\quad f(x) = \ln(x+1) \qquad x = 0$ (this is a)

$f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{x+1} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(x+1)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{2}{(x+1)^3} \qquad f'''(0) = 2$

$f^{(4)}(x) = \frac{-3\cdot 2}{(x+1)^4} \qquad f^{(4)}(0) = -3!$

$$P_4 = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 3\frac{x^4}{4!} + \quad \text{Taylor Series}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \qquad \sum_{n=1}^{\infty} \frac{x^n(-1)^n}{n}$$

This series converges for $-1 < x \leq 1$

So we can't simply go on and on. We will have to stop somewhere

$$f(x) = P_n(x) + R_n(x)$$

where P_n - n^{th} degree polynomial
R_n - remainder

Recall from MVT Calc I

there is a c in (a,b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

let $b = x$ so $$\frac{f(x) - f(a)}{x - a} = f'(c)$$

or $$f(x) = \underbrace{f(a)}_{P_0(x)} + \underbrace{f'(c)(x-a)}_{R_0(x)}$$

if we replace $f \to f'$

so $f'(x) = f'(a) + f''(c)(x-a)$

now $\int$ $f(x) = f'(a)x + \frac{f''(c)(x-a)^2}{2!} + k$

to find k sub $x=a$, so

$$f(a) = f'(a)a + \frac{f''(c)(a-a)^2}{2} + k$$

$$k = f(a) - af'(a)$$

so
$$f(x) = f'(a)x + \frac{f''(c)(x-a)^2}{2!} + f(a) - af'(a)$$

$$= \underbrace{f(a) + f'(a)(x-a)}_{P_1(x)} + \underbrace{\frac{f''(c)(x-a)^2}{2!}}_{R_1(x)}$$

so we keep going in this fashion

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)(x-a)^n}{n!}}_{P_n} + \underbrace{\frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}}_{R_n}$$

So the Taylor Remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$ where c is between a & x

Tomorrow we'll get an idea on how to use this

Ex $f(x) = e^x$ find P_3 & R_3 at $x = 0$

so we'll need 3 derivatives for P_3 & another for R_3

$f(x) = e^x \quad f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

$f'''(x) = e^x \quad f'''(0) = 1$

$f^{(4)}(x) = e^x$ for R

$P_3 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad R_3 = \frac{e^c x^4}{4!}$ c between 0 & x

Ex $f(x) = \cos x$ find P_4 R_4 when $x = \frac{\pi}{4}$

$f(x) = \cos x$ $\quad f(\pi/4) = \frac{1}{\sqrt{2}}$

$f' = -\sin x$ $\quad f'(\pi/4) = -\frac{\sqrt{3}}{2}$

$f'' = -\cos x$ $\quad f''(\pi/4) = -\frac{\sqrt{2}}{2}$

$f''' = \sin x$ $\quad f'''(\pi/4) = \frac{\sqrt{2}}{2}$

$f^{(4)} = \cos x$ $\quad f^{(4)}(\pi/4) = \frac{\sqrt{2}}{2}$

$f^{5} = -\sin x$ Remainder

$$P_4 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^2}{2!} + \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{3!} + \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^4}{4!}$$

$$R_4 = -\sin c \frac{(x - \pi/4)^5}{5!}$$

Consider

$f(x) = \sin x \qquad P_4(x) \quad \text{at } x = a$

$f = \sin x$
$f' = \cos x$
$f'' = -\sin x$
$f''' = -\cos x$
$f^{(4)} = \sin x$
$f^{(5)} = \cos x$

$$P_4 = x - \frac{x^3}{3!} \qquad R_4 = \cos c \, \frac{x^5}{5!}$$

so how accurate is this?

$$P_4(.1) = .1 - \frac{.1^3}{3!} = .0998\dot{3}$$

let's add more terms

$$P_6(.1) = .1 - \frac{.1^3}{3!} + \frac{.1^5}{5!} = .09983341\dot{6}$$

$$P_8 = .1 - \frac{.1^3}{3!} + \frac{.1^5}{5!} - \frac{.1^7}{7!} = .099833341664$$

so by adding terms the answer is getting better

so using 3 terms the answer is accurate to 8 digits

$$\sin .1 \simeq .09983341\cancel{7}$$

So going out

$$P_4 = x - \frac{x^3}{3!} \qquad R_4 = \cos c \, \frac{x^5}{5!}$$

$$P_6 = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad R_6 = -\cos c \, \frac{x^7}{7!}$$

$$P_8 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \qquad R_8 = \cos c \, \frac{x^9}{9!}$$

in general could we examine

$$R_{2n} = \pm \cos c \, \frac{x^{2n+1}}{(2n+1)!} \quad \leftarrow \text{this}$$

to see how far we can go out

to get a desired accuracy?