Calculus 3 - Optimization

Critical Points

The point (a, b) is called a critical point if

1. $f_x(a, b) = 0$ and $f_y(a, b) = 0$
2. $f_x(a, b)$ or $f_y(a, b)$ DNE

Second Derivative Test

$$\Delta = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b) \tag{1}$$

1. If $\Delta > 0$ and $f_{xx}(a, b) > 0$ then f has a relative minimum at (a, b)
2. If $\Delta > 0$ and $f_{xx}(a, b) < 0$ then f has a relative maximum at (a, b)
3. If $\Delta < 0$ then f has a saddle at (a, b)
4. If $\Delta = 0$ test fails

We now consider two examples.

Example 1.

Find the dimensions of the rectangular box with a fixed volume of 64 inch^3 with minimal surface area.

Soln. We first draw and label the box with sides $x, y,$ and z.

The volume of the box is

$$V = xyz = 64. \tag{2}$$

The surface area of the box is

$$A = 2xy + 2xz + 2yz. \tag{3}$$

Solving (2) for z gives

$$z = \frac{64}{xy}. \tag{4}$$

and thus

$$\begin{aligned} A &= 2xy + 2x \cdot \frac{64}{xy} + 2y \cdot \frac{64}{xy} \\ &= 2xy + \frac{2 \cdot 64}{y} + \frac{2 \cdot 64}{x}. \end{aligned} \tag{5}$$

This is what we wish to minimize. Taking first partial derivatives gives

$$A_x = 2y - \frac{2 \cdot 64}{x^2}, \quad A_y = 2x - \frac{2 \cdot 64}{y^2}. \tag{6}$$

Setting these to zero gives

$$2y - \frac{2 \cdot 64}{x^2} = 0, \quad 2x - \frac{2 \cdot 64}{y^2} = 0. \tag{7}$$

From the first

$$y = \frac{64}{x^2} \tag{8}$$

and substituting into the second gives

$$x - \frac{64}{\left(\frac{64}{x^2}\right)^2} = 0 \quad \Rightarrow \quad x - \frac{x^4}{64} = 0 \quad \Rightarrow \quad x^3 = 64 \tag{9}$$

and so $x = 4$ and this gives $y = \frac{64}{4^2} = \frac{64}{16} = 4$. Now we use the second derivative test. Calculating second order partial derivatives

$$A_{xx} = \frac{2 \cdot 2 \cdot 64}{x^3}, \quad A_{xy} = 2, \quad A_{yy} = \frac{2 \cdot 2 \cdot 64}{y^3} \tag{10}$$

and so

$$\Delta = A_{xx}A_{yy} - A_{xy}^2 = \frac{2 \cdot 2 \cdot 64}{x^3} \cdot \frac{2 \cdot 2 \cdot 64}{y^3} - 4 \tag{11}$$

and at the critical point

$$\Delta = \frac{2 \cdot 2 \cdot 64}{4^3} \cdot \frac{2 \cdot 2 \cdot 64}{4^3} - 4 = 12 > 0 \tag{12}$$

and $A_{xx} > 0$ so we have a minimum. Finally, $z = 64/(xy) = 64/16 = 4$ so the dimensions are $4'' \times 4'' \times 4''$.

Example 2.

Find the minimum distance from the origin to the plane $2x + y + 2z = 9$.

Soln. We use the distance formula is

$$s = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{13}$$

where (x, y, z) is a point on the plane (x_0, y_0, z_0) a given point and since we are at the origin then $(x_0, y_0, z_0) = (0, 0, 0)$ and so (13) is

$$s = \sqrt{x^2 + y^2 + z^2}. \tag{14}$$

Now we bring in the plane so

$$y = 9 - 2x - 2z \tag{15}$$

so (14) becomes

$$s = \sqrt{x^2 + (9 - 2x - 2z)^2 + z^2}, \tag{16}$$

To minimize s is to minimize s^2 so we set $S = s^2$

$$S = x^2 + (9 - 2x - 2z)^2 + z^2, \tag{17}$$

Calculating first partials gives

$$S_x = 2x - 4(9 - 2x - 2z), \quad S_z = -4(9 - 2x - 2z) + 2z \tag{18}$$

$$= 10x + 8z - 36 \qquad\qquad = 8x + 10z - 36 \tag{19}$$

These we set to zero so

$$10x + 8z - 36 = 0, \quad 8x + 10z - 36 = 0 \tag{20}$$

We certainly could solve one of these for x (or z) and sub. into the other but instead I will subtract these two giving $z = x$. Thus

$$10x + 8x - 36 = 0 = 0 \quad \Rightarrow \quad 18x - 36 = 0 \quad \Rightarrow \quad x = 2 \tag{21}$$

so $x = z = 2$. Now we use the second derivative test. Calculating second order partials gives

$$S_{xx} = 10, \quad S_{xy} = 8, \quad S_{yy} = 10, \tag{22}$$

and so

$$\Delta = S_{xx}S_{yy} - S_{xy}^2 = 10 \cdot 10 - 8^2 > 0 \tag{23}$$

and since $S_{xx} > 0$ we have a min. Finally, since $2x + y + 2z = 9$ the point on the plane is $(2, 1, 2)$ and the distance from the origin to this point is $s = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$.