Math 4315 - PDEs Monday Oct. 17/16

2nd order PDE's

$$a u_{xx} + b u_{xy} + c u_{yy} + \text{lots} = 0$$

Last class we hit modified hyperbolic form

$$u_{rs} + \text{lots} = 0$$

We achieved this by finding r & s that satisfy

$$a r_x^2 + b r_x r_y + c r_y^2 = 0 \qquad a s_x^2 + b s_x s_y + c s_y^2 = 0$$

Ex $u_{xx} - 4x^2 u_{yy} + 2u_y = 0$

$b^2 - 4ac = 16x^2 > 0$ if $x \neq 0$ so hyp.

so $r_x^2 - 4x^2 r_y^2 = 0 \qquad s_x^2 - 4x^2 s_y^2 = 0$

$(r_x - 2x r_y)(r_x + 2x r_y) = 0$ similar for s

Pick $r_x - 2x r_y = 0 \qquad s_x + 2x s_y = 0$

M of C $r = R(x^2 + y) \qquad s = S(x^2 - y)$

Pick $r = x^2 + y \qquad s = x^2 - y$

Chain Rule

$r_x = 2x,\ r_y = 1,\quad r_{xx} = 2\quad r_{xy} = r_{yy} = 0$

$s_x = 2x\quad s_y = -1\quad s_{xx} = 2\quad s_{xy} = s_{yy} = 0$

$u_y = u_r - u_s$

$u_{xx} = 4x^2 u_{rr} + 8x^2 u_{rs} + 4x^2 u_{ss} + 2u_r + 2u_s$

$u_{xy} = 2x\, u_{rr} + 0\, u_{rs} - 2x\, u_{ss}$

$u_{yy} = u_{rr} - 2u_{rs} + u_{ss}$

sub gives $\quad 16x^2 u_{rs} + 4u_r = 0 \quad$ or $\quad u_{rs} + \dfrac{u_r}{4x^2} = 0$

From the change of variables

$r = x^2 + y \quad s = x^2 - y \Rightarrow r + s = 2x^2$

so target is $\quad u_{rs} + \dfrac{u_r}{2(r+s)} = 0$

So how do we get to the regular hyp SF

$u_{ss} - u_{rr} + \text{lots} = 0$

Consider $\quad u_{xx} - u_{yy} = 0 \quad$ which is hyperbolic

to hits MHSF $\quad u_{rs} + \text{lots} = 0$

we need to solve

$$r_x^2 - r_y^2 = 0 \qquad s_x^2 - s_y^2 = 0$$

$$r_x - r_y = 0 \qquad s_x + s_y = 0$$

$$r = R(x+y) \qquad s = S(x-y)$$

so $r = x+y$, $s = x-y$

this goes from $u_{xx} - u_{yy} = 0 \longrightarrow u_{rs} = 0$

to go the other way

$$x = \frac{r+s}{2}, \quad y = \frac{r-s}{2}$$

but the division by 2 really doesn't matter as it puts a factor into the eq^n that can be cancelled. So can we get to the change of variables right away? Yes!

Recall if we have $ax^2+bx+c=0$
if $b^2-4ac \geq 0$ then ↗ has 2 factor. Can use quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ so the +/− separates

just like $x, y = \frac{r \pm s}{2}$

This is how we find the change of variable

Ex
$$u_{xx} - 3u_{xy} + 2u_{yy} = 0$$

$$r_x^2 - 3r_x r_y + 2r_y^2 = 0$$

Note: $r_y \neq 0$ because if so $r_x = 0$ & $r = $ const

then $\left(\frac{r_x}{r_y}\right)^2 - 3\frac{r_x}{r_y} + 2 = 0$ quadratic in $\frac{r_x}{r_y}$

$$\frac{r_x}{r_y} = \frac{3 \pm \sqrt{3^2 - 4.1.2}}{2} = \frac{3 \pm 1}{2}$$

Now follow the +/- through

$$2r_x = (3 \pm 1) r_y$$

$$2r_x - (3 \pm 1) r_y = 0$$

M of C $\quad \frac{dx}{2} = \frac{dy}{-(3 \pm 1)}$; $dr = 0$ $\quad (3 \pm 1)dx = -2dy$

$$(3 \pm 1)x + 2y = c_1 \quad c_2 = r$$

$$r = R\left((3\pm 1)x + 2y\right)$$

$$= R\left(3x + 2y \pm x\right)$$

Pick $\quad r = 3x + 2y, \quad s = x \qquad$ (or vice versa if you wish)

$$U_{xx} = 9U_{rr} + 6U_{rs} + U_{ss}$$

$$U_{xy} = 6U_{rr} + 2U_{rs}$$

$$U_{yy} = 4U_{rr}$$

sub $\quad U_{xx} - 3U_{xy} + 2U_{yy} = 0$

$$\Rightarrow \quad 9U_{rr} + \cancel{6U_{rs}} + U_{ss}$$

$$- 3(6U_{rr} + \cancel{2U_{rs}})$$

$$+ 2 \cdot 4U_{rr} = 0$$

$$(9 - 18 + 8)U_{rr} + U_{ss} = 0$$

$$\Rightarrow \quad -U_{rr} + U_{ss} = 0 \qquad \text{target form}$$

Ex2 $\quad x^2 u_{xx} - xy\, u_{xy} - 2y^2 u_{yy} = 1 \qquad$ (Note the 1)

This is hyperbolic $\because\ b^2 - 4ac > 0$

$$x^2 r_x^2 - xy\, r_x r_y - 2y^2 r_y^2 = 0$$

$$\Rightarrow \quad x^2\left(\frac{r_x}{r_y}\right)^2 - xy\frac{r_x}{r_y} - 2y^2 = 0 \Rightarrow \quad \frac{r_x}{r_y} = \frac{xy \pm \sqrt{x^2y^2 + 8x^2y^2}}{2x^2}$$

so $\displaystyle \frac{r_x}{r_y} = \frac{xy \pm 3xy}{2x^2} = \frac{(1\pm3)y}{2x}$

$$\Rightarrow \quad 2x\, r_x - (1\pm3)y\, r_y = 0 \quad \text{so} \quad \frac{dx}{2x} = \frac{-dy}{(1\pm3)y}\ ;\ dr = 0$$

$$\Rightarrow \quad (1\pm3)\frac{dx}{x} = -2\frac{dy}{y} \quad \Rightarrow \quad (1\pm3)\ln x + 2\ln y = c_1, \quad c_2 = r$$

$$\Rightarrow \quad r = R\left(\ln x + 2\ln y \pm 3\ln x\right)$$

Pick $\quad r = 3\ln x, \quad s = \ln x + 2\ln y$

so $\quad \displaystyle U_{xx} = \frac{9U_{rr}}{x^2} + \frac{6U_{rs}}{x^2} + \frac{U_{ss}}{x^2} - \frac{3U_r}{x^2} - \frac{U_s}{x^2}$

$$U_{xy} = \frac{6U_{rs}}{xy} + \frac{2U_{ss}}{xy}$$

$$U_{yy} = \frac{4U_{ss}}{y^2} - \frac{2U_s}{y^2}$$

PDE $x^2 u_{xx} - xy u_{xy} - 2y^2 u_{yy} = 1$

$\Rightarrow$ $9U_{rr} + 6\cancel{U_{rs}} + U_{ss} - 3U_r - U_s$

$- 6\cancel{U_{rs}} - 2U_{ss} - 8U_{ss} + 4U_s = 1$

$9U_{rr} - 9U_{ss} - 3U_r + 3U_s = 1$

$\Rightarrow$ $U_{rr} - U_{ss} - \frac{1}{3} U_r + \frac{1}{3} U_s = \frac{1}{9}$

OR $U_{ss} - U_{rr} + \frac{1}{3} U_r - \frac{1}{3} U_s = -\frac{1}{9}$

either of those are fine!

Next class - will target

$u_{rr} + u_{ss} + (\text{lots} = 0$