Math 1497 - Calc 2

Infinite Series

we test whether $\sum_{n=1}^{\infty} a_n$, $a_n > 0$

converges

Test #1 nth term test

If $\lim_{n\to\infty} a_n = \#$ (not 0) the series div

Ex 1 $\sum_{n=1}^{\infty} \frac{n+1}{2n}$ $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} \neq 0$ so by the nth term test the series div

Test #2 $\int$ test

Here we consider $\int_1^{\infty} f(n)\,dn$ where $f(n) = a_n$

Conditions (i) f cont^s, (ii) $f \geq 0$ (iii) f dec

then test applies

Ex 2 $\sum_{n=1}^{\infty} n e^{-n^2}$ Here $f(n) = n e^{-n^2}$

f is cont's ✓ & > 0 ✓ $f' = e^{-n^2} - 2n^2 e^{-n^2}$

$= (1-2n^2)\, e^{-n^2} < 0$ ✓

so test applies

$$\lim_{b\to\infty} \int_1^b n e^{-n^2}\,dn = \lim_{b\to\infty} -\tfrac{1}{2} e^{-n^2} \Big|_1^b = \lim_{b\to\infty} -\tfrac{1}{2} e^{-b^2} + \frac{e^{-1}}{2}$$

$= \frac{e^{-1}}{2}$ so by the $\int$ test our series converges

(converges)

<u>Test #3</u> <u>LCT</u>

we compare $\sum_{n=1}^{\infty} a_n$ $\sum_{n=1}^{\infty} b_n$

if $\lim_{n\to\infty} \frac{a_n}{b_n} = \#$ (not 0) both series do the same

ex3 $\sum \frac{n}{\sqrt{n^4+1}}$ compare w/ $\frac{n}{\sqrt{n^4}} = \frac{n}{n^2} \to \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^4+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+1}} = \lim_{n\to\infty} \frac{n^2}{n^2\sqrt{1+\frac{1}{n^4}}} = 1$$

$\because \sum \frac{1}{n}$ div then by LCT our series div.

Test #4 DCT

we again compare series using the inequality

$$0 < a_n < b_n$$

if $\sum a_n$ div then $\sum b_n$ div

if $\sum b_n$ conv then $\sum a_n$ conv.

Ex4 $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

$\because$ $0 < \sin^2 n < 1$ then $0 < \frac{\sin^2 n}{n^3} < \frac{1}{n^3}$

$\because$ $\sum \frac{1}{n^3}$ conv $(p=3)$ by DCT our series conv

Test 5 Ratio Test

if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ if $L > 1$ series div

$L < 1$ series conv

$L = 1$ no conclusion

Ex5 $\sum \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \Big/ \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

so by ratio test the series conv.

Test #6 Root Test

If $\lim_{n \to \infty} \sqrt[n]{a_n} = L$

$L > 1$ series div

$L < 1$ series conv

$L = 1$ no conclusion

Ex 6 $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

$$\lim_{n \to \infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n \to \infty} \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}}$$

$= \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n} = e > 1$ so by root test the series div

Now we consider series when the terms go negative as well as positive

One such series is an "alternating series"

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

Now $\sum \frac{1}{n}$ div but what about this series

let's group the terms by 2

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)\cdots\left(\frac{1}{n}-\frac{1}{n+1}\right)\cdots$$

$$\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots\frac{1}{n(n+1)}$$

so we could write the series as

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} \text{ this conv (LCT w/ } \sum \frac{1}{n^2})$$

so we ~~have~~ the last test the alternating series Test

#7 AST

If $\sum_{n=1}^{\infty}(-1)^n a_n$ a $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ $a_n > 0$

If (1) $\lim_{n\to\infty} a_n = 0$

(2) $a_{n+1} < a_n$ dec

the series will converge.

Previous Ex

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \qquad a_n = \frac{1}{n}$$

(i) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ (ii) $f = \frac{1}{n}$ $f' = \frac{-1}{n^2} < 0$ dec

So by AST the series conv ✓

<u>Ex 7</u> $\sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n}$

$$a_n = \frac{\ln n}{n} \qquad \lim_{n\to\infty} \frac{\ln n}{n} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{n} = 0 \checkmark$$

$$f(n) = \frac{\ln n}{n} \qquad f' = \frac{n \cdot \frac{1}{n} - \ln n}{n^2} = \frac{1 - \ln n}{n^2} < 0$$

if $n > e$

So by AST the series converges

Absolute & conditional convergence

we consider $\sum_{n=1}^{\infty} (-1)^n a_n$

if $\sum a_n$ converges then $\sum (-1)^n a_n$ converges absolutely

if $\sum a_n$ diverges but $\sum (-1)^n a_n$ converges by AST

then we say $\sum (-1)^n a_n$ converges conditimly

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\quad \sum \frac{1}{n^2}$ converges $p=2$

so ↑ converges absolutely

ex $\sum \frac{(-1)^n}{n}$ $\quad \sum \frac{1}{n}$ div by $\sum \frac{(-1)^n}{n}$ conv (already shown)

so ↑ converges conditionally

let's look at some sums as we go out in the series

N	$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$	$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2}$
100	.688172	.822418
1000	.692647	.822467
10,000	.693097	.822467
100,000	.693142	
1,000,000	.693146	

Note: Given $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

if we stop at $n = N$ the error is at most a_{N+1}

$\sum \frac{(-1)^n}{n}$ $a_n = \frac{1}{n}$ $a_{1,000,001} = \frac{1}{1,000,001} = 1 \times 10^{-6}$

$\sum \frac{(-1)^n}{n^2}$ $a_n = \frac{1}{n^2}$ $a_{10,001} = \frac{1}{(10,001)^2} = 1 \times 10^{-8}$

$$\nabla a + \left(\frac{\cos 2b - \sin 2b}{a^2-2}\right) \cdot \nabla b + \left(\frac{\cos 2b + \sin 2b - a}{a^2-2} - \frac{H'}{H}\right) \nabla b = 0$$

$$\nabla a \quad \left(\frac{\cos 2b + \sin 2b + a}{a^2-2} + \frac{H'}{H}\right) \nabla b - \left(\frac{\cos 2b - \sin 2b}{a^2-2}\right) \nabla b = 0$$

Define $\frac{F}{a^2-2} = \frac{H'}{H} + \frac{a}{a^2-2}$

$$(a^2-2)\nabla a + (\cos 2b - \sin 2b)\nabla b + (\cos 2b + \sin 2b - F)\nabla b = 0$$

$$(a^2-2)\nabla a + (\cos 2b + \sin 2b + F)\nabla b - (\cos 2b - \sin 2b)\nabla b = 0$$

Add

$$(a^2-2)\nabla a + (a^2-2)\nabla a + (2\cos 2b + F)\nabla a$$

$$+ (2\sin 2b - F)\nabla b$$