Math 4381/6378 Symmetry

Last class we considered the symmetries of $u_t = u_x^2$ and found them to be

$$T = c_1 x^2 + (2c_2 t + c_3) x + 4c_4 t^2 + c_5 t + c_6$$

$$X = -2(2c_1 x + 2c_2 t + c_3) u + c_2 x^2 + (4c_4 t + c_7) x + 2c_8 t + c_9$$

$$U = -4c_1 u^2 + (2c_2 x - c_5 + 2c_7) u - c_4 x^2 - c_8 x + c_{10}$$

So we ask - is it possible to use these to solve say

$$u_t = u_x^2 \quad \text{subject to} \quad u(x,0) = x^2 + x$$

Since we created the invariant surface condition

$$T u_t + X u_x = U$$

the sol$^n$ should also satisfy this.

This means the boundary condition should as well

so if $u(x,0) = x^2 + x$

$u_x(x,0) = 2x+1$

& from the PDE $u_t = u_x^2 = (2x+1)^2$

so we sub $t=0$ and these ↑ into the ISC

so

$$(c_1x^2 + c_3x + c_6)(2x+1)^2$$

$$+ \left[ -2(2c_1x + c_3)(x^2+x) + c_2x^2 + c_7x + c_9 \right] [2x+1]$$

$$= -4c_1(x^2+x)^2 + (2c_2x - c_5 + 2c_7)(x^2+x)$$

$$- c_4x^2 - c_8x + c_{10}$$

we then compare coefficient

we require that

$$c_6 + c_9 - c_{10} = 0$$

$$-c_3 + c_8 + c_5 + c_5 + 4c_6 - c_7 + 2c_9 = 0$$

$$c_1 - 2c_3 - c_2 + c_4 + c_5 + 4c_6 = 0$$

so we obtain

$$c_5 = c_2 + 2c_3 - 4c_6 - c_1 - c_4$$

$$c_7 = c_3 + c_8 + c_2 - c_1 - c_4 + 2c_9$$

$$c_{10} = c_6 + c_9$$

so we have a fairly extensive set of ISC to deal with so we consider

For example $c_1 = 1$ others $= 0$

$$(x^2 - t)\, u_t - x(4u+1)\, u_x = -u(4u+1)$$

M of C
$$\frac{dt}{x^2 - t} = \frac{dx}{-x(4u+1)} = \frac{du}{-u(4u+1)}$$

(1) $\dfrac{dx}{x} = \dfrac{du}{u} \Rightarrow \dfrac{u}{x} = c_1$

(2) $\dfrac{dt}{x^2 - t} = \dfrac{dx}{-x(4u+1)}$ so $\dfrac{dt}{x^2 - t} = \dfrac{dx}{-x(4c_1 x + 1)}$

so $\dfrac{dt}{dx} = \dfrac{x^2 - t}{-x(49x+1)}$ linear

$$\frac{dt}{dx} - \frac{t}{x(49x+1)} = \frac{-x}{49x+1}$$

integrating fact $\mu = \dfrac{49x+1}{x}$

so $\dfrac{d}{dx}\left(\dfrac{49x+1}{x} . t\right) = -1$

$\Rightarrow \quad \dfrac{49x+1}{x} . t = -x + c_2$

so $\quad \dfrac{4\cancel{x}\frac{y}{\cancel{x}}+1}{x} . t + x = c_2$

sol^n $\quad \dfrac{4u+1}{1} . \dfrac{t}{x} + x = F\left(\dfrac{y}{x}\right)$

$$u_t = \frac{4u+1}{F' - 4t}, \quad u_x = \frac{uF' + x^2 - 4tu - t}{x(F' - 4t)^2}$$

and sub into $u_t = u_x^2$ and use the sol$^n$ to obtain

$$r^2 F'^2 - 2rFF' + F^2 - F' = 0$$ a nonlinear ODE

we will return there in a moment.

Next, we try another symmetry

$\underline{\underline{g_6}}$ $\quad (1-4t)u_t = 4u+1$

$$\frac{dt}{1-4t} = \frac{du}{4u+1}$$

$-\ln|1-4t| = \ln 4u+1 - \ln g$

$x = c_2$

$$4u+1 = \frac{f(x)}{1-4t} \Rightarrow 4u = \frac{f(x)}{1-4t} - 1$$

or $$u = \frac{f(x) - 1 + 4t}{4(1-4t)}$$

sub into $u_t = u_x^2$

$$f'^2 = 16f \qquad \frac{df}{\sqrt{f}} = 4\,dx$$

$$2\sqrt{f} = 4x + 2k$$

$$\Rightarrow \quad f = (2x+k)^2$$

so $$u = \frac{(2x+k)^2 + 4t - 1}{4(1-4t)}$$

Now for $k$? $\quad u(x,0) = x^2 + x$

$$x^2 + x = \frac{(2x+k)^2 - 1}{4}$$

$$4x^2 + 4x = 4x^2 + 4kx + k^2 - 1 \quad \leftarrow \text{choose } k = 1$$

$$u = \frac{(2x+1)^2 + 4t - 1}{4(1-4t)}$$

$$= \frac{4x^2 + 4x + \cancel{1} + 4t - \cancel{1}}{4(1-4t)} = \frac{x^2 + x + t}{1-4t}$$

exact sol$^n$

Maybe there's more info we can obtain

Let's return to the a symm.

ISC $(x^2-t)u_t - x(4u+1)u_x = -u(4u+1)$

Sol$^n$ $\dfrac{(4u+1)t}{x} + x = F\left(\dfrac{u}{x}\right)$

Reduction $r^2F'^2 - 2rFF' + F^2 - F' = 0$ *

IC $u(x,0) = x^2+x$ sub into sol$^n$

$x = F\left(\dfrac{x^2+x}{x}\right) = F(x+1)$ let $r=x+1$

$F(r) = r-1$ (does satisfy *)

So $\dfrac{4u+1}{x}\cdot t + x = \dfrac{u}{x} - 1$

$4ut + t + x^2 = u - x$

$x^2+x+t = (1-4t)u \Rightarrow u = \dfrac{x^2+x+t}{1-4t}$