Sample Test 1 - Solutions

1. Find the unit tangent and unit normal vector for the following vector functions

$$(i)\quad \vec{r}(t) = < 2t, t^2 >$$
$$(ii)\quad \vec{r}(t) = < e^t \cos t, e^t \sin t >$$

Sol(i)

$$\begin{aligned} \overrightarrow{r} &= \left\langle 2t, t^2 \right\rangle \\ \overrightarrow{r}' &= \langle 2, 2t \rangle \\ \|\overrightarrow{r}'\| &= 2\sqrt{t^2+1}. \end{aligned}$$

so

$$\overrightarrow{T} = \frac{\overrightarrow{r}'}{\|\overrightarrow{r}'\|} = \left\langle \frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}} \right\rangle$$

Further

$$\begin{aligned} \overrightarrow{T}' &= \left\langle \frac{-t}{(t^2+1)^{3/2}}, \frac{1}{(t^2+1)^{3/2}} \right\rangle \\ \|\overrightarrow{T}'\| &= \frac{1}{t^2+1}. \end{aligned}$$

so

$$\overrightarrow{N} = \frac{\overrightarrow{T}'}{\|\overrightarrow{T}'\|} = \left\langle \frac{-t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle$$

Sol(ii)

$$\begin{aligned} \overrightarrow{r} &= \left\langle e^t \cos t, e^t \sin t \right\rangle \\ \overrightarrow{r}' &= \left\langle e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t \right\rangle \\ \|\overrightarrow{r}'\| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2}\, e^t. \end{aligned}$$

so

$$\overrightarrow{T} = \frac{\overrightarrow{r}'}{\|\overrightarrow{r}'\|} = \left\langle \frac{\cos t - \sin t}{\sqrt{2}}, \frac{\sin t + \cos t}{\sqrt{2}} \right\rangle$$

Further

$$\overrightarrow{T}' = \left\langle \frac{-\sin t - \cos t}{\sqrt{2}}, \frac{\cos t - \sin t}{\sqrt{2}} \right\rangle.$$

since

$$\|\overrightarrow{T}'\| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1,$$

then

$$\overrightarrow{N} = \frac{\overrightarrow{T}'}{\|\overrightarrow{T}'\|} = \left\langle \frac{-\sin t - \cos t}{\sqrt{2}}, \frac{\cos t - \sin t}{\sqrt{2}} \right\rangle.$$

2. Prove the limits either exist or do not exist. In the former case use the squeeze theorem.

$$(i) \lim_{(x,y)->(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2} \qquad (ii) \lim_{(x,y)->(0,0)} \frac{y - x^3}{y + x^3}$$

$$(iii) \lim_{(x,y)->(0,0)} \frac{x^3 + y^3}{x^2 + y^2} \qquad (iv) \lim_{(x,y)->(0,0)} \frac{x^4 + 2y^4}{x^2 + y^2}$$

2 (i)

$$\text{Along} \quad y = 0, \lim_{(x,y)->(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2} = \lim_{(x,y)->(0,0)} \frac{x^2}{x^2} = 1$$

$$\text{Along} \quad y = x, \lim_{(x,y)->(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2} = \lim_{(x,y)->(0,0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

Since following different paths lead to different limits, the limit DNE.

2 (ii)

$$\text{Along} \quad y = 0, \lim_{(x,y)->(0,0)} \frac{y - x^3}{y + x^3} = \lim_{(x,y)->(0,0)} \frac{-x^3}{x^3} = -1$$

$$\text{Along} \quad x = 0, \lim_{(x,y)->(0,0)} \frac{y - x^3}{y + x^3} = \lim_{(x,y)->(0,0)} \frac{y}{y} = 1.$$

Since following different paths lead to different limits, the limit DNE.

2 (iii) From the inequalities

$$-\sqrt{x^2 + y^2} \leq x \leq \sqrt{x^2 + y^2}$$
$$-\sqrt{x^2 + y^2} \leq y \leq \sqrt{x^2 + y^2}$$

we have

$$-\left(x^2 + y^2\right)^{3/2} \leq x^3 \leq \left(x^2 + y^2\right)^{3/2}$$
$$-\left(x^2 + y^2\right)^{3/2} \leq y^3 \leq \left(x^2 + y^2\right)^{3/2}$$

which gives

$$-2\left(x^2 + y^2\right)^{3/2} \leq x^3 + y^3 \leq 2\left(x^2 + y^2\right)^{3/2}.$$

Thus,

$$-2\left(x^2+y^2\right)^{1/2} \leq \frac{x^3+y^3}{x^2+y^2} \leq 2\left(x^2+y^2\right)^{1/2}$$

and

$$-2 \lim_{(x,y)->(0,0)} \left(x^2+y^2\right)^{1/2} \leq \lim_{(x,y)->(0,0)} \frac{x^3+y^3}{x^2+y^2} \leq 2 \lim_{(x,y)->(0,0)} \left(x^2+y^2\right)^{1/2}.$$

Since

$$\lim_{(x,y)->(0,0)} \left(x^2+y^2\right)^{1/2} = 0$$

by the squeeze theorem

$$\lim_{(x,y)->(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0.$$

2 (iv) From the inequalities

$$0 \leq x^2 \leq x^2+y^2$$
$$0 \leq y^2 \leq x^2+y^2$$

we have

$$0 \leq x^4 \leq \left(x^2+y^2\right)^2$$
$$0 \leq y^4 \leq \left(x^2+y^2\right)^2$$

which gives

$$0 \leq x^4+2y^4 \leq 3\left(x^2+y^2\right)^2.$$

Thus,

$$0 \leq \frac{x^4+2y^4}{x^2+y^2} \leq 3\left(x^2+y^2\right)$$

and

$$0 \leq \lim_{(x,y)->(0,0)} \frac{x^4+2y^4}{x^2+y^2} \leq 3 \lim_{(x,y)->(0,0)} x^2+y^2.$$

Since

$$\lim_{(x,y)->(0,0)} x^2+y^2 = 0$$

by the squeeze theorem

$$\lim_{(x,y)->(0,0)} \frac{x^4 + 2y^4}{x^2 + y^2} = 0.$$

3. Find the equation of the tangent plane to the given surface at the specified point

$$x^2y + xz + yz^2 = 3, \quad P(1, 2, -1)$$

Sol: If we define $F = x^2y + xz + yz^2 - 3$ then $F_x = 2xy + z$, $F_y = x^2 + z^2$ and $F_z = x + 2yz$. Evaluating these at the point P gives $F_x = 3$, $F_y = 2$ and $F_z = -3$. The equation of the tangent plane is thus $3(x-1) + 2(y-2) - 3(z+1) = 0$

4. If $z = x^2 - y^2$, calculate the following chain rules:

$(i)\ \frac{dz}{dt}$ if $x = \cos t$, and $y = \sin t$

$(ii)\ \frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ if $x = \frac{\cos s}{r}$, and $y = \frac{\sin s}{r}$

Sol: We will need the following derivatives

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = -2y, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dx}{dt} = \cos t,$$
$$\frac{\partial x}{\partial r} = -\frac{\cos s}{r^2}, \quad \frac{\partial x}{\partial s} = -\frac{\sin s}{r}, \quad \frac{\partial y}{\partial r} = -\frac{\sin s}{r^2}, \quad \frac{\partial y}{\partial s} = \frac{\cos s}{r}.$$

$$(i) \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x\left(-\sin t\right) - 2y\left(\cos t\right) = -4\sin t\cos t.$$

$$(ii) \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2x\left(-\frac{\cos s}{r^2}\right) - 2y\left(-\frac{\sin s}{r^2}\right) = 2\,\frac{\sin^2 s - \cos^2 s}{r^3}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2x\left(-\frac{\sin s}{r}\right) - 2y\left(\frac{\cos s}{r}\right) = \frac{4\sin s\cos s}{r^2}$$

5. Find the directional derivative of $z = x^2 + 3xy + y^2$ at $(1, 1)$ in the direction of $< -3, 4 >$. In what direction should you move for maximum increase?

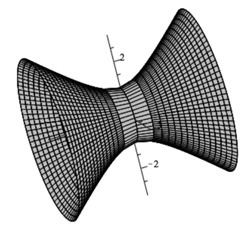
Sol: The gradient is given by $\nabla z =< 2x + 3y, 3x + 2y >$ and at the point $(1, 1)$ it becomes $\nabla z =< 5, 5 >$. The direction derivative is then given by

$$\nabla z \cdot \frac{\vec{u}}{\|\vec{u}\|} = < 5, 5 > \cdot \frac{< -3, 4 >}{5} = \frac{-15 + 20}{5} = 1$$

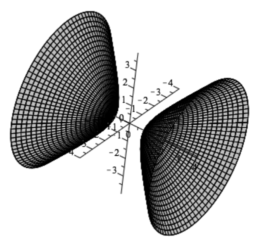
The directional you should travel for maximum increase is in the direction of the gradient $< 5, 5 >$.

6. Sketch and name the following surface

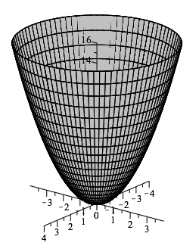
$(i)\quad -x^2 + y^2 + z^2 = 1$ Hyperboloid of 1 sheet



$(ii)\quad -x^2 + y^2 - z^2 = 1$ Hyperbolid of 2 sheets



$(iii)\quad x^2 + y^2 - z = 0$ Paraboloid



$(iv)\quad -y^2 + z = 0$ Parabolic cylinder