Math 6345 - AODEs

Lyapunov functions & Stability

Ex 1

$$\dot{x} = y + kx(x^2+y^2)$$
$$\dot{y} = -x + ky(x^2+y^2)$$

Consider $V = x^2+y^2$ (this was r^2)

so $V(0,0) = 0$ $V(x,y) > 0$ if $(x,y) \neq (0,0)$

so $\dot{V} = 2x\dot{x} + 2y\dot{y}$

$$= 2x\left(y + kx(x^2+y^2)\right) + 2y\left(-x + ky(x^2+y^2)\right)$$

$$= 2kx^2(x^2+y^2) + 2ky^2(x^2+y^2)$$

$$= 2k(x^2+y^2)^2$$

& $\dot{V} > 0$ if $k > 0$ so unstable

$\dot{V} < 0$ if $k < 0$ so Asy. stable

Ex 2 $\dot{x} = -x - 2y^2$

$\dot{y} = xy - y^3$

Try $V = x^2 + y^2$ (already shown to be Lypunov)

$\dot{V} = 2x\dot{x} + 2y\dot{y}$

$= 2x(-x - 2y^2) + 2y(xy - y^3)$

$= -2x^2 - 4xy^2 + 2xy^2 - 2y^4$

$= -2x^2 - 2xy^2 - 2y^4$

↑ close - would like this to be gone

Next, try $V = x^2 + ay^2$ let's find a

if $a > 0$ then V is Lyp

$\dot{V} = 2x\dot{x} + 2ay\dot{y}$

$= 2x(-x - 2y^2) + 2ay(xy - y^3)$

$= -2x^2 - 4xy^2 + 2axy^2 - 2ay^4$ pick $a = 2$

$\dot{V} = -2x^2 - 4y^4 < 0$ so $(0,0)$ is Asy. stable

Ex 3 $\quad \dot{x} = -x^3 + xy^2$

$\dot{y} = -2x^2y - y^3$

CP $\quad x(-x^2-y^2) = 0 \quad y(2x^2+y^2) = 0$

$y=0$ or $(0,0)$

$y = 0 \Rightarrow x = 0$

so $(0,0)$ is only CP.

$$D_x f = \begin{pmatrix} -3x^2+y^2 & 2xy \\ -4xy & -2x^2-3y^2 \end{pmatrix} \qquad D_x f\big|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$V = ax^2 + by^2 \quad (a, b > 0)$

$\dot{V} = 2ax\dot{x} + 2by\dot{y}$

$= 2ax(-x^3 + xy^2) + 2by(-2x^2y - y^3)$

$= -2ax^4 + 2ax^2y^2 - 4bx^2y^2 - 2by^4$

choose $a = 2b$

$\dot{V} = -4bx^4 - 2by^4 < 0$ if $b > 0$

so $(0,0)$ AS.

Ex4 $\dot{x} = x - 2y^3$

$\dot{y} = x + y + x^2 y$

CP $x = 2y^3$ so $2y^3 + y + 4y^7 = 0$

$y(1 + 2y^2 + 4y^6) = 0 \Rightarrow y = 0$ only

so $(0, 0)$

Linear System

$$D_x f = \begin{pmatrix} 1 & -6y^2 \\ 1 + 2xy & 1 + x^2 \end{pmatrix}$$

$$D_x f\Big|_{cp} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\dot{\bar{x}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \bar{x}$$

Linear System predicts unstable

Try

$V = x^2 + y^2$

$\dot{V} = 2x\dot{x} + 2y\dot{y}$

$= 2x(x - 2y^3) + 2y(x + y + x^2 y)$

$= 2x^2 - 4xy^3 + 2xy + 2y^2 + 2x^2y^2 \quad \begin{matrix} > 0 \\ < 0 \end{matrix} \; ?$

Now Try

$V = x^2 + y^4$

$\dot{V} = 2x\dot{x} + 4y^3\dot{y}$

$= 2x(x - 2y^3) + 4y^3(x + y + x^2 y)$

$= 2x^2 - \cancel{4xy^3} + \cancel{4xy^3} + 4y^4 + 4x^2y^4$

$= 2x^2 + 4y^4(x^2+1) > 0$

away from $(0,0)$

So Lypunov also predicts unstable origin