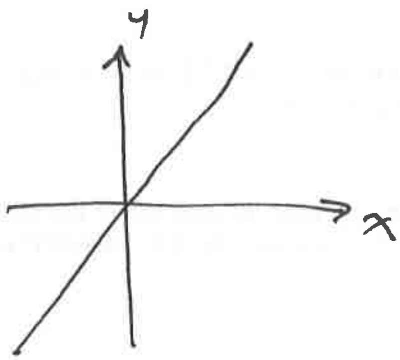
Math 4315 – PDE's

Fourier Series – Half Interval

Before considering a Fourier Series on $[0, L]$ we need to discuss odd & even fcts.

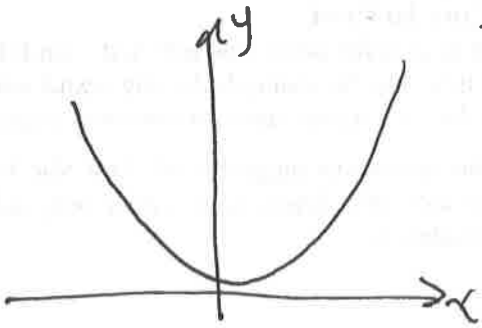
Consider $f(x) = x$



this is an odd function

mean $f(-x) = -f(x)$

and $F(x) = x^2$



this is an even function

meaning

$f(-x) = f(x)$

we can see $\displaystyle\int_{-L}^{L} x\,dx = \frac{x^2}{2}\Big|_{-L}^{L} = \frac{L^2}{2} - \frac{L^2}{2} = 0$

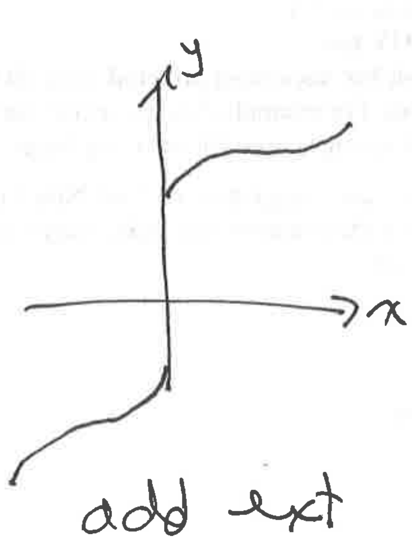
& $\displaystyle\int_{-L}^{L} x^2\,dx = \frac{x^3}{3}\Big|_{-L}^{L} = \frac{2L^3}{3} = 2\int_{0}^{L} x^2\,dx$

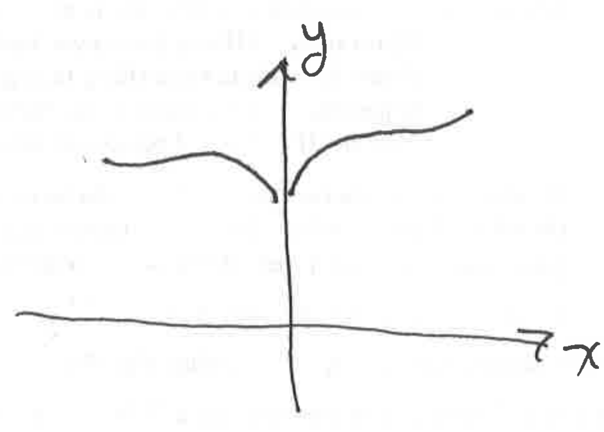
This is true of all odd & even functions

if $f(x)$ is odd $\int_{-L}^{L} f(x)dx = 0$

if $f(x)$ is even $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$

so now we construct a Fourier series when $f(x)$ is given on $[0, L]$. We first extend $f(x)$ to $[-L, 0]$. We can do an odd extension or even extension





Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x .$$

where $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

if $f(x)$ is odd (the extension)

then $f(x)\cos\frac{n\pi}{L}x$ is "odd · even = odd" so $a_n = 0$

$f(x)\sin\frac{n\pi}{L}x$ is "odd · odd = even"

so $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \qquad a_n = 0$

if $f(x)$ is even (the extension)

then $f(x)\cos\frac{n\pi}{L}x$ is "even . even = even"

$f(x)\sin\frac{n\pi}{L}x$ is "even. odd = odd"

$\Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \qquad b_n = 0$

The following examples illustrate

Fourier Cosine Series

If $f(x)$ has an even extension

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$$

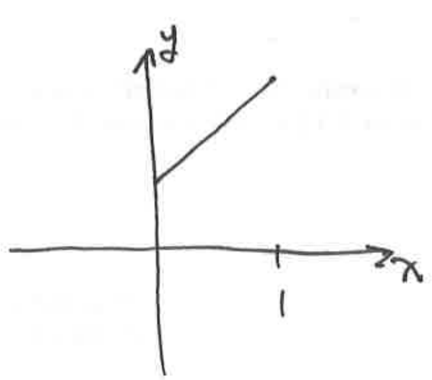
Fourier Sine Series

If $f(x)$ has an odd extension

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$

Ex 1 $f(x) = x+1$ on $[0,1)$



Fourier Cosine Series

$$a_0 = \frac{2}{1}\int_0^1 (x+1)\,dx = 2\left(\frac{x^2}{2}+x\Big|_0^1\right) = 3$$

$$a_n = \frac{2}{1}\int_0^1 (x+1)\cos n\pi x\,dx$$

$$= 2\left(\frac{\cos n\pi x + n\pi x \sin n\pi x}{n^2\pi^2}\Big|_0^1 + \frac{\sin n\pi x}{n\pi}\Big|_0^1\right)$$

$$= 2\left[\left(\frac{\cos n\pi + n\pi \cancel{\sin n\pi}^{\,0}}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right) + \frac{\cancel{\sin n\pi}^{\,0}}{n\pi} - 0\right]$$

$$= 2\cdot\frac{\cos n\pi - 1}{n^2\pi^2}$$

$$f(x) = \frac{3}{2} + 2\sum_{n=1}^{\infty}\frac{\cos n\pi - 1}{n^2\pi^2}\cos n\pi x$$

Fourier Sine Series

$$b_n = \frac{2}{1}\int_0^1 (x+1)\sin n\pi x \, dx$$

$$= 2\left[\frac{\sin n\pi x - n\pi x \cos n\pi x}{n^2\pi^2}\Bigg|_0^1 - \frac{\cos n\pi x}{n\pi}\Bigg|_0^1 \right]$$

$$= 2\left[\frac{\sin \overset{0}{\cancel{n\pi}} x - n\pi \cos n\pi x}{n^2\pi^2} - \frac{(\cos n\pi - 1)}{n\pi} \right]$$

$$= 2 \cdot \frac{1 - 2\cos n\pi}{n\pi}$$

$$f(x) = 2\sum_{n=1}^{\infty} \frac{1-2\cos n\pi}{n\pi} \cdot \sin n\pi x$$