PDE's

1st order

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

chain Rule

$$u_s = u_x x_s + u_y y_s$$

so CE $\quad x_s = a, \; y_s = b \quad u_s = c$

Procedure
1) solve with 3 aub fcts
2) eliminate s
3) eliminate r

Ex $\quad u_x + y u_y = 1$

$x_s = 1 \Rightarrow x = s + a(r) \quad \leftarrow \quad s = x - a(r)$

$y_s = y \Rightarrow y = b(r)e^s \qquad y = b(r)e^{x-a(r)}$

$u_s = 1 \qquad u = s + c(r) \qquad = b(r)e^{-a(r)}e^x$

$\qquad = D(r)e^x$

$u - x = c(r) - a(r)$

$\quad = E(r)$

$u - x = E\, D^{-1}(y e^{-x})$

$$\boxed{u = x + f(e^{-x}y)}$$

ex 2 $u_x + u u_y = 0$

so CE $x_s = 1 \Rightarrow x = s + a(r)$

$y_s = u \Rightarrow y_s = c(r) \Rightarrow y = c(r)s + b(r)$

$u_s = 0 \Rightarrow u = c(r)$

$y = c(r)(x - a(r)) + b(r)$

$= c(r)x - a(r)c(r) + b(r)$

$y = xu + D(r)$

so $y - xu = D(c^{-1}(u))$

$y - xu = f(u)$

or $r = D^{-1}(y - xu)$

$u = c(D^{-1}(y - xu))$

$u = g(y - xu)$

both work

Now what about the f? IC.

Solve

$u_x + 2u_y = -1 \qquad u(x,0) = 3x$

$x_s = 1 \Rightarrow x = s + a(r)$

$y_s = 2 \Rightarrow y = 2s + b(r)$

$2x - y = 2a(r) - b(r) = D(r)$

$u_s = -1 \Rightarrow u = -s + c(r) \qquad u + x = a + c = E(r)$

$u + x = E(D^{-1}(2x-y))$

$u = -x + f(2x-y)$

Now bring in IC

when $y = 0 \quad u = 3x$

$3x = -x + f(2x - 0)$

$f(2x) = 4x$

let argument $2x = \lambda$

so $f(\lambda) = 2\lambda$

$u = -x + 2(2x-y)$

$\boxed{u = 3x - 2y}$

check IC 1st

$u(x,0) = 3x - 0 = 3x$ ✓

Solve $xu_x - yu_y = u \quad u(x,x) = 1$

CE $\quad x_s = x \Rightarrow x = A(r)e^{s}$

$y_s = -y \qquad y = B(r)e^{-s}$

$u_s = u \qquad u = C(r)e^{s}$

(1) $\quad xy = A(r)\cancel{e^{s}}B(r)\cancel{e^{-s}} = D(r)$

(2) $\quad \frac{u}{x} = \frac{C(r)\cancel{e^{s}}}{A(r)\cancel{e^{s}}} = E(r)$

so $\quad \frac{u}{x} = f(xy) \Rightarrow \quad u = x\,f(xy)$ $\quad$ sol'n

B.C. $\quad y = x, \; u = 1$

so $\quad 1 = x\,f(x \cdot x) \Rightarrow \quad f(x^2) = \frac{1}{x}$

let $\lambda = x^2 \quad x = \sqrt{\lambda} \quad \Rightarrow \quad f(\lambda) = \frac{1}{\sqrt{\lambda}}$

so $\quad u = x\frac{1}{\sqrt{xy}} = \frac{\sqrt{x}}{\sqrt{y}}$ $\qquad$ check $u(x,x) = \frac{\sqrt{x}}{\sqrt{x}} = 1$ ✓

$u_x = \frac{1}{2\sqrt{x}\sqrt{y}}, \quad u_y = -\frac{\sqrt{x}}{2y\sqrt{y}}$

$xu_x - yu_y = \frac{\sqrt{x}}{2\sqrt{y}} - \left(-\frac{\sqrt{x}}{2\sqrt{y}}\right) = u$ ✓