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Research Article

Analysis of Convex Optimization and Applications to Financial Engineering

S. O. Asamba, C. M. Kaunda, N. B. Okelo*

School of Mathematics and Actuarial Science,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya.

*Corresponding author's e-mail: bnyaare@yahoo.com

Abstract

Convex optimization has become more interesting in studies because of its efficient applications in finance, management science, automatic control systems, economics, signal and image processing, statistics and data analysis. In the present work, we give an in depth analysis of convex optimization in Banach spaces. Lastly, we give the applications to financial engineering.

Keywords: Semi-continuous function; Convexity; Optimization; Financial engineering; Hilbert space.

Introduction

Optimization which is also known as Mathematical programming entails maximizing or minimizing a desired objective function while ensuring that the prevailing constraints (inequalities, equalities or abstract constraints) are satisfied [1]. Optimization is an important field of Mathematics owing to its efficient applications in economics, management science, automatic control systems, engineering finance, electronic circuit design, communication and networks. Despite optimization problems having known to exist for a long time, research on optimization theory field autonomously from classical calculus has been done in the last few decades.

Euler's Calculus of variations gave rise to what today is known as analytic optimization. Lagrange came up with the concept of Lagrange Multipliers which are the auxiliary variables from which optimality conditions are derived. The author in [2] was the first to obtain results on convex sets after Jensen had introduced the notion of convex functions. He investigated calculus derivatives and the concept of maxima and minima [3]. This led to publishing of the first optimization textbook by Hancock which was titled, Theory of Minima and Maxima of functions of several variables [4]. In [5] the researcher studied optimal economic growth which he later renamed 'the optimal growth theory' and he was the first to give applications of calculus of variations to economics.

Two remarkable developments in optimization were done during the time of the Second World War. The first was invention of the Simplex method algorithm for solving linear programming problems by [6].This algorithm was very useful in planning and decision making in large scale enterprises. The second was establishment of the theory of duality for linear programming problems by Von Neumann. In [7] the author developed interior point methods for linear programming which were used in operations research and engineering. More research proved that interior point methods can efficiently solve some classes of convex problems, semi definite programs and second order cone programs as easily as linear programs. Research on optimization up to late 1940's was focused on calculus of variations and solving of linear programming problems.

Significant research on optimization from 1950's has focused more on non-linear programming. This was given a boost by the invention of optimality conditions for non-linear problems by Kuhn and Tucker similar to optimality conditions earlier given by [8]. These optimality conditions are today called the Kurush-Kuhn-Tucker (KKT) optimality conditions.

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Optimality conditions of non-linear optimization problem were investigated using a class of generalized convex sets and convex functions called E-convex functions. Majeed and In [9] the author did not consider convex optimization in L^p spaces nor characterization of semi-continuous functions. Some researchers like in [10] have studied optimization in Hilbert spaces. The article [1] dealt with minimization of convex functionals on infinite dimensional real Hilbert spaces. This was optimization of convex functionals in Hilbert spaces.

The study [5] considered global optimization in Hilbert space. They developed a complete-search algorithm for solving non-convex infinite dimensional optimization problems in Hilbert space. Houska and Chachuat [9] considered non-convex optimization problems in Hilbert spaces. In [3] they investigated interpolating curve or surface with linear inequality constraints posing it as a general convex optimization problem in a finite dimensional Hilbert space. They showed that the approximate solution converges uniformly to the optimal constrained interpolating function. An algorithm was derived to produce such solutions. This study was focused only at interpolating in a convex subset of a Hilbert space but not general convex optimization in L^p spaces. In [6] the author investigated convex optimization problems in infinite dimensional real Hilbert spaces and provided a first order optimality condition for convex optimization problems. Optimization of Convex functionals in infinite dimensional real Hilbert spaces is discussed in detail and an efficient application to Dirichlet optimization problem is given. The main properties of general normed spaces especially the linear functionals and topological dual were investigated. Some concepts of convex optimization in Hilbert spaces like projection and orthogonality were also investigated.

Preliminaries

In this section, we start by reviewing the basic definitions and results on: Convex sets, convex functions, and semi-continuous functions, global and local minimizers.

Definition 2.1

An inner product on a vector space V *is* a map $\langle .,. \rangle : V \times V \to \mathbb{K}$ such that $\forall x, y, z \in V$ and $\lambda \in \mathbb{K}$ the following are satisfied:

(i). $\langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0$, *if and only if* $x = 0$.

(ii). $\langle x+y, z \rangle = \langle x,z \rangle + \langle y,z \rangle$

(iii). $\langle \lambda x, y \rangle = \lambda \langle x,y \rangle$

(iv). $\langle x,y \rangle = \overline{\langle y,x \rangle}$

The ordered pair $(V, \langle .,. \rangle)$ is called an inner product space.

Definition 2.2

A nonnegative real valued function $\|.\| : V \to \mathbb{R}$ is called a norm if it satisfies the following properties:

(i) $\| x \| \geq 0$ *and* $\| x \| = 0$ iff $x = 0$ $\forall x \in V$

(ii) $\| x + y \| \leq \| x \| + \| y \|$ $\forall x, y \in V$

(iii) $\| \lambda x \| = |\lambda| \, \| x \|$ $\forall x \in V, \lambda \in \mathbb{C}$

The ordered pair $(V, \|.\|)$ is called a normed space.

Definition 2.3

A Banach space is a complete normed linear space.

Definition 2.4

A Hilbert space H *is* a complete inner product space

Definition 2.5

A sequence x_n in a Banach space B is said to converge to $x \in B$ if $lim_{n \to \infty} x_n = x$. A sequence x_n in a Hilbert space H converges weakly to x if, $\lim_{n \to \infty} \langle x_n, u \rangle = \langle x,u \rangle$, < $\forall u \in B$.

Definition 2.6

A function f is with the domain X is said to be continuous at a point $a \in X$, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that $\forall x \in X, |f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$. If the function f is continuous at each point of the domain X then f is said to be continuous on X.

Definition 2.7

A real valued function $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be lower semi-continuous (lsc) if $f(X) \leq \liminf_{n\to\infty} f(x_n)$ for all sequences $\{x_n\} \in \mathbb{R}^n$ such that $x_n \to x\ (strongly)\ \forall x \in \mathbb{R}^n$.

Definition 2.8

A real valued function $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be upper semi-continuous (usc) if $\forall x \in \mathbb{R}^n$, $f(X) \geq \limsup_{n\to\infty} f(x_n)$ for all sequences $\{x_n\} \in \mathbb{R}^n$.

Definition 2.9

A non-empty set V *is* said to be convex if for all $x, y \in V, \lambda \in [0,1]$ implies $x + (1-\lambda)y \in V$. Let X be a metric space and $V \subseteq X$ be a non-empty set. A function $f : V \to \mathbb{R}$ is convex if $\forall\lambda \in [0,1$ and $\forall x, y \in V, f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

Remark 2.10

A function f is convex if and only if its epigraph (the set of points on or above the graph of the function) denoted as $epi\, f$ is convex, where the epigraph of f is defined by

$$epi\, f = \{f(x,r) \in dom(f) \times R : r \geq f(x)\}.$$

Definition 2.11

Let $V \subseteq R^n$ where $\mathbb{R}^n$ is an n-dimensional real space. A point $x^* \in \mathbb{R}^n$ is a global minimizer of the optimization problem $min_{x\in V} f(x)$, if $x^* \in V$ and $f(x^*) \leq f(x), \forall x \in V$.

Definition 2.12

Let $V \subseteq \mathbb{R}^n$ where $\mathbb{R}^n$ is an n-dimensional real space. A point $x^* \in \mathbb{R}^n$ is called a *local minimizer* of the optimization problem $min_{x \in V} f(x)$, if there exists a neighborhood N of x^* such that x^* is a global minimizer of the problem $min_{x\in V\cap N} f(x)$. That is, there exists $\varepsilon > 0$ such that $f(x^*) \leq f(x)$, whenever $x^* \in V$ satisfies $\| x^* - x \| \leq \varepsilon$.

Remark 2.13

An optimization problem is convex if both the feasible set and the objective function are convex. Moreover, every local minimizer of a convex optimization problem is a global minimizer.

Definition 2.14

Let (X, A, β) be a measure space and $1 \leq p < \infty$. The space $L^p(X)$ consists of equivalence classes of measurable functions $f: X \to \mathbb{R}$ such that $\int |f|^p d\beta < \infty$, where two measurable functions are are equivalent if they are equal $\beta - a$.e. The L^p-norm of $f \in L^p(X)$ is defined by

$$\| f \|_{L^p} = \left(\int |f|^p d\beta\right)^{\frac{1}{p}}.$$

Results and discussions

In this section, we give the main results of our study. We begin with the following theorem.

Theorem 3.1

Let f be an arbitrary function from $\mathbb{R}^n$ to $[-\infty, +\infty]$, then the following conditions are equivalent:

a) f is lower semi-continuous throughout $\mathbb{R}^n$

b) $x\, f(x) \leq \alpha$ is closed for every $\alpha \in \mathbb{R}$

c) The epigraph of f is closed set in $\mathbb{R}^{n+1}$

Theorem 3.1 shows the importance of lower semi-continuity in the study of convex functions. The convex functions were in infinite dimensional real spaces and not in L^p spaces.

Theorem 3.2

Let C1 and C2 be non-empty sets in $\mathbb{R}^n$. There exists a hyper plane separating C1 and C2 properly if and only if there exists a vector b such that

a) $\inf\{\langle x,b\rangle : x \in C_1\} \geq \sup\{\langle x,b\rangle : x \in C_2\}$,

b) $\sup\{\langle x,b\rangle : x \in C_1\} > \inf\{\langle x,b\rangle : x \in C_2\}$.

There exists a hyper plane separating C1 and C2 strongly if a vector b such that

c) $\inf\{\langle x,b\rangle : x \in C_1\} > \sup\{\langle x,b\rangle : x \in C_2\}$.

Theorem 3.2 investigates proper separation and strong separation of convex sets in infinite dimensional real spaces and not convex optimization in L^p spaces.

Theorem 3.3

A closed convex set C is the intersection of the closed half spaces which contain it.

Proof

It suffices to show that every closed convex set in R^n can be represented as some solution set of a system of weak linear inequalities. The convex sets in infinite dimensional real spaces were characterized and not convex optimization in L^P spaces.

Corollary 3.14

A non-empty closed convex set containing no lines has at least one extreme point. In [7] the author investigated and cameup with results on convex analysis but his work was limited to the infinite dimensional space of all n-tuples of real numbers (i.e $\mathbb{R}^n$) only and not convex optimization in L^P spaces.

Proof

In [6] the researchers solved portfolio optimization problems with linear transaction costs, diversification constraints and limits on variance and on shortfall risk by casting them as convex optimization problems and obtaining their global solutions by use of a relaxation method which easily obtains a computable upper bound through convex optimization. Portfolio optimization in this case involves seeking the best way to invest some capital in a set of n assets. They considered a model problem in which variable x_i was used to represent the investment in the i^{th} asset, so the vector $x \in \mathbb{R}^n$ described the overall portfolio allocation across the set of assets. The constraints were a limit on the budget (i.e., a limit on the total amount to be invested), the requirement that investments are non-negative (assuming short positions are not allowed), and a minimum acceptable value of expected return for the whole portfolio. The objective or cost function was to be a measure of the overall risk or variance of the portfolio return. In this case, the optimization problem was to choose a portfolio allocation that minimizes risk, among all possible allocations that meet the firm requirements. They also solved larger non convex portfolio optimization problems by solving a small number of convex optimization problems hence yielding suboptimal portfolio and an upper bound on the global optimum. This study used convex optimization on portfolio problems with linear and fixed transaction costs but did not study convex optimization in L^P spaces.

Remark 3.5

In [4] investigated convex optimization problems in infinite dimensional spaces. Duality approach was used to solve the optimization problems. Conditions necessary for duality formalism were developed such that they require the optimal value of the convex optimization problem to vary continuously with respect to perturbations in the constraints only along feasible directions hence implying the existence of the dual problem. These conditions were posed as certain local compactness requirements on the dual feasibility set based on characterization of locally compact convex sets in locally convex spaces in terms of non-empty relative interiors of the corresponding polar sets. Optimization tools included convex analysis and the theory of conjugate functions.

Theorem 3.6

Let X be a Hausdorff locally Convex Space (HLCS), $f: X \to \overline{\mathbb{R}}$ convex and M an affine subset of X with the induced topology, $M \supset \mathrm{dom} f$. Let f (.) be bounded above on a subset C of X where $\mathrm{ri}\, C = \emptyset$ and $\mathrm{af}\, f\, C$ is closed with finite co-dimension in M. Then rcorcodom f $= \emptyset$, cof restricted to rcorcodom f is continuous and af f dom f is closed with finite co-dimension in M. Moreover, $f^{\sim} \equiv +\infty$ if there exists $x_0 \in X, r_0 > -f(x_0)$ such that $\{y \in X * f*(y) - \langle x_0, y\rangle\} \leq r_0$ is $w(X^*, X)/M^{\perp}$ locally bounded.

Theorem 3.7

Let C be a convex, strongly closed and bounded subset of H. Suppose $f: C \to R$ is a strongly lower semi-continuous and convex function. Then f is bounded from below and attains a minimizer on C. In [2], the researchers developed a complete-search algorithm for solving non-convex infinite dimensional optimization problems in Hilbert space. They also constructed a global optimization algorithm whose worse-case run-time complexity is independent of the number of optimization variables and the algorithm remained tractable for infinite dimensional optimization problems.

The study of [3] investigated general convex optimization problems and came up with a new method for interpolating in a convex subset of a Hilbert space. They considered interpolating curve or surface with linear inequality constraints as convex problems in a reproducing kernel Hilbert space. An approximation method based on a discretized optimization problem in a finite-dimensional Hilbert space was introduced under the same set of constraints. They proved that the approximate solution converges uniformly to the optimal constrained interpolating function. An algorithm for solving such optimization problems was also developed.

In [8] they studied classical results on optimization of convex functionals in Hilbert spaces. Tools of convex analysis and lower semi-continuous functions are used. Sufficient proofs for classical theorems of convex optimization are given. Authors presented a first order optimality condition and gave a detailed illustration of an application of convex optimization involving Dirichlet problem. The main results obtained include:

Theorem 3.8

Let H be an infinite dimensional real Hilbert space and $W \subseteq H$ be weakly sequentially closed and bounded set. Let $f: W \to R$ be weakly sequentially lsc. Then f is bounded from below and has a minimizer on W.

Applications

Semi-continuity and convex optimization have been investigated by many researchers being motivated by needs in finance, economics, engineering, automatic control systems, management science, and statistics. Semi-continuity and convex optimization in L^p spaces have not been exhaustively investigated. Characterization of semi-continuous functions and results of investigation of convex optimization in L^p spaces will have exemplary applications in finance, particularly in giving efficient solutions to portfolio selection, asset allocation, option pricing, risk management, asset and liability management.

Conclusions

In the present paper, we have studied the notion of convex optimization that has become more interesting in recent studies because of its efficient applications in finance, management science, automatic control systems, economics, signal and image processing, statistics and data analysis. In particular, we give an in-depth analysis of convex optimization in Banach spaces. Lastly, we have given the applications to financial engineering

Conflict of interest

Authors have declared no conflict of interests.

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