Jacobians and transformation of variables

Chain rule

Consider

$$u = r^2 s \tag{1}$$

and the change of variables

$$r = x^2 + y^2, \qquad s = xy. \tag{2}$$

Eliminating r and s in (1) gives

$$u = xy\left(x^2 + y^2\right)^2. \tag{3}$$

If we wish to calculate partial derivatives with respect to x and y then we would simply calculate the partial derivatives directly obtaining

$$u_x = y\left(x^2 + y^2\right)^2 + 4x^2 y\left(x^2 + y^2\right), \tag{4a}$$

$$u_s = x\left(x^2 + y^2\right)^2 + 4xy^2\left(x^2 + y^2\right). \tag{4b}$$

In calculus III, we were introduced to the chain rule for functions of two variables, namely

$$u_x = u_r r_x + u_s s_x, \tag{5a}$$

$$u_y = u_r r_y + u_s s_y \tag{5b}$$

For the preceding example

$$\begin{array}{ll} u_r = 2rs, & u_s = r^2, \\ r_x = 2x, & r_y = 2y, \\ s_x = y, & s_y = x. \end{array}$$

and from (5)

$$\begin{aligned} u_x &= 2rs \cdot 2x + r^2 \cdot y \\ &= 4y\left(x^2 + y^2\right)^2 + 4x^2 y\left(x^2 + y^2\right), \\ u_s &= 2rs \cdot 2y + r^2 \cdot x, \\ &= x\left(x^2 + y^2\right)^2 + 4xy^2\left(x^2 + y^2\right), \end{aligned}$$

giving exactly (4). Consider $u = u(x, y)$ (now unkown) and the change of variables

$$r = x + y, \qquad s = x - y. \tag{6}$$

To calculate u_x and u_y we again use (5) giving

$$u_x = u_r + u_s, \tag{7a}$$

$$u_y = u_r - u_s. \tag{7b}$$

If the change of variables is

$$r = 2xy, \qquad s = x^2 + y^2. \tag{8}$$

then u_x and u_y become

$$u_x = u_r \cdot 2y + u_s \cdot 2x, \tag{9a}$$

$$u_y = u_r \cdot 2x + u_s \cdot 2y. \tag{9b}$$

However, in (9) the variables x and y still exist and (8) would need to be used to put everything in terms of r and s. In this case

$$x = \frac{1}{2}\left(\sqrt{s+r} + \sqrt{s-r}\right), \quad y = \frac{1}{2}\left(\sqrt{s+r} - \sqrt{s-r}\right) \tag{10}$$

giving

$$u_x = \left(\sqrt{s+r} - \sqrt{s-r}\right) u_r + \left(\sqrt{s+r} + \sqrt{s-r}\right) u_s, \tag{11a}$$

$$u_y = \left(\sqrt{s+r} + \sqrt{s-r}\right) u_r + \left(\sqrt{s+r} - \sqrt{s-r}\right) u_s. \tag{11b}$$

Suppose the change of variables is

$$x = r\cos s, \qquad y = r\sin s. \tag{12}$$

To calculate u_x and u_y (from (5)) we would need to solve (12) for r and s. We can of course do this ($r = \sqrt{x^2 + y^2}$ and $s = \tan^{-1} y/x$) but things are getting complicated. If the change of variables is

$$x = r^2 + s^2 r^3, \qquad y = s + r^2 s^3 + s^5, \tag{13}$$

then we have a problem as we can't solve (13) for r and s.

A natural question is – Is there a way to find these derivatives without solving say (12) or (13) for (r, s) explicitly? The answer is yes! We use Jacobians. The Jacobian $J(u, v, x, y)$ is defined as

$$J(u, v, x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}. \tag{14}$$

In terms of Jacobians, we can write the derivatives u_x and u_y as

$$u_x = \frac{\partial(u, y)}{\partial(x, y)}, \qquad u_y = \frac{\partial(x, u)}{\partial(x, y)}, \tag{15}$$

and then use the chain rule for Jacobians, *i.e.*

$$u_x = \frac{\partial(u, y)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \frac{\dfrac{\partial(u, y)}{\partial(r, s)}}{\dfrac{\partial(x, y)}{\partial(r, s)}}, \tag{16}$$

or

$$u_y = \frac{\partial(x, u)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \frac{\dfrac{\partial(x, u)}{\partial(r, s)}}{\dfrac{\partial(x, y)}{\partial(r, s)}}, \tag{17}$$

and these are easier to calculate. For example, for the change of variables (12), substituting (12) in (16) and (17) gives

$$u_x = \frac{\begin{vmatrix} u_r & u_s \\ y_r & y_s \end{vmatrix}}{\begin{vmatrix} x_r & x_s \\ y_r & y_s \end{vmatrix}} = \frac{\begin{vmatrix} u_r & u_s \\ \sin s & r\cos s \end{vmatrix}}{\begin{vmatrix} \cos s & -r\sin s \\ \sin s & r\cos s \end{vmatrix}} = \frac{r\cos s u_r - \sin s u_s}{r} \tag{18}$$

and

$$u_y = \frac{\begin{vmatrix} x_r & x_s \\ u_r & u_s \end{vmatrix}}{\begin{vmatrix} x_r & x_s \\ y_r & y_s \end{vmatrix}} = \frac{\begin{vmatrix} \cos s & -r\sin s \\ u_r & u_s \end{vmatrix}}{\begin{vmatrix} \cos s & -r\sin s \\ \sin s & r\cos s \end{vmatrix}} = \frac{r\sin s u_r + \cos s u_s}{r} \tag{19}$$

so

$$u_x = \cos s\, u_r - \frac{\sin s}{r} u_s, \quad u_y = \sin s\, u_r + \frac{\cos s}{r} u_s. \tag{20}$$

Two things to note: (1) no need to solve for r and s and (2) the derivatives are all in terms of r and s.