Math 4315 - PDEs

Ordinary Differential Equations Review - Part 2

1 Linear Systems

A linear system of equations

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \tag{1}$$

can be can be written as a matrix ODE

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{2}$$

where $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we consider solutions of the form

$$\bar{x} = \bar{c}e^{\lambda t},$$

then after substitution into (2) we obtain

$$\lambda\bar{c}\, e^{\lambda t} = A\,\bar{c}\, e^{\lambda t}$$

from which we deduce

$$(\lambda I - A)\,\bar{c} = 0. \tag{3}$$

In order to have nontrivial solutions $\bar{c}$, we require that

$$|\lambda I - A| = 0. \tag{4}$$

This is the eigenvalue-eigenvector problem. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then (4) becomes

$$\lambda^2 - (a+d)\lambda + a\,d - b\,c = 0,$$

from which we have three cases:

1. two distinct eigenvalues
2. two repeated eigenvalues,
3. two complex eigenvalues.

Here we consider an example of the first, two distinct eigenvalues.

Example 1 If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \tag{5}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda - 1 & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$

From (3) we have

$$\begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$

From (3) we have

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to (5) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$

Alternate Form

Sometimes a system of ODEs can be written as

$$\frac{dt}{P(t,x,y)} = \frac{dx}{Q(t,x,y)} = \frac{dy}{R(t,x,y)}. \tag{6}$$

This is similar to the alternate form for a single ODE

$$\frac{dy}{dx} = F(x,y) \text{ or } M(x,y)dx + N(x,y)dy = 0.$$

One could write (6) in terms of the usual system

$$\frac{dx}{dt} = \frac{Q}{P}, \quad \frac{dy}{dt} = \frac{R}{P},$$

and determine whether its linear or nonlinear and proceed as above but sometimes its not possible nor desirable. Consider the following

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{du}{3u}.$$

Here, it is easier to pick them in pairs, say for example

$$\frac{dx}{x} = \frac{dy}{2y}, \quad \frac{dx}{x} = \frac{du}{3u}.$$

Each are easily solved giving rise to

$$\frac{y}{x^2} = c_1, \quad \frac{u}{x^3} = c_2.$$

Example 2 Consider

$$\frac{dx}{u-x} = \frac{dy}{2x} = \frac{du}{u-x}. \tag{7}$$

Here we need to be somewhat choosy in how we pick our pairs as not all pairs will work (*i.e.* a pair with only two variables). The choice here is the first and third

$$\frac{dx}{u-x} = \frac{du}{u-x},$$

as this simplifies to

$$dx = du,$$

which integrate to $u = x + c_1$. With this we substitute into the original system and obtain

$$\frac{dx}{c_1} = \frac{dy}{2x} = \frac{dx}{c_1}.$$

noting that we now in fact have only a single pair

$$\frac{dx}{c_1} = \frac{dy}{2x}.$$

Upon integration, we obtain

$$x^2 = c_1 y + c_2,$$

and using c_1 obtained previously, we get

$$x^2 = (u-x)y + c_2.$$

Example 3 Consider

$$\frac{dx}{x} = \frac{dy}{x+y} = \frac{dz}{x+y+z}. \tag{8}$$

Again, choose wisely. Here we choose the first pair

$$\frac{dx}{x} = \frac{dy}{x+y}, \quad \text{or} \quad \frac{dy}{dx} = \frac{x+y}{x}$$

which we find as its solution

$$\frac{y}{x} = \ln|x| + c_1. \tag{9}$$

The next pair in (8) is

$$\frac{dx}{x} = \frac{dz}{x+y+z}$$

or

$$\frac{dz}{dx} = \frac{x+y+z}{x}$$

which, upon expanding, becomes

$$\frac{dz}{dx} = 1 + \frac{y}{x} + \frac{z}{x},$$

which is linear in z. Using (9) gives

$$\frac{dz}{dx} = 1 + \ln|x| + c_1 + \frac{z}{x},$$

Integrating gives

$$\frac{z}{x} = \frac{1}{2}\ln^2|x| + (c_1+1)\ln|x| + c_2,$$

and eliminating c_1 gives

$$\frac{z}{x} = \frac{1}{2}\ln^2|x| + \left(\frac{y}{x} - \ln|x| + 1\right)\ln|x| + c_2.$$

Example 4 Consider

$$\frac{dx}{y+z} = \frac{dy}{y} = \frac{dz}{x-y}. \tag{10}$$

Here it is impossible to choose a pair that only involves 2 variables so we need to be very clever. Consider the first and second as a pair and the second and third terms as a pair and re-write as

$$\frac{dx}{dy} = \frac{y+z}{y}, \quad \frac{dz}{dy} = \frac{x-y}{y}. \tag{11}$$

Now here's the clever part, add and subtract the two ODEs in (11)

$$\frac{d(x+z)}{dy} = \frac{x+z}{y}, \quad \frac{d(x-z)}{dy} = \frac{2y-x+z}{y}. \tag{12}$$

If we let $u = x + z$ and $v = x - z$, then (12) becomes

$$\frac{du}{dy} = \frac{u}{y}, \quad \frac{dv}{dy} = \frac{2y - v}{y},$$

from which we find the solution

$$\frac{u}{y} = c_1, \quad y\,v = y^2 + c_2,$$

or, in term of the original variables

$$\frac{x+z}{y} = c_1, \quad (x - z) - y^2 = c_2.$$

Example 5 Consider

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{1} = \frac{dp}{2p} = \frac{dq}{2q}. \tag{13}$$

Here, there a 5 independent variables x, y, u, p, and q. Again, we pick in pairs. First we pick only the first two in (13)

$$\frac{dx}{x} = \frac{dy}{y}, \tag{14}$$

and obtain the solution $\frac{y}{x} = c_1$ or $y = c_1 x$. Using this in (13) gives

$$\frac{dx}{x} = \frac{c_1 dx}{c_1 x} = \frac{du}{1} = \frac{dp}{2p} = \frac{dq}{2q}, \tag{15}$$

noting the first two terms in (15) are identical (after cancellation) and thus we really only have

$$\frac{dx}{x} = \frac{du}{1} = \frac{dp}{2p} = \frac{dq}{2q}, \tag{16}$$

Now we pick another pair $-$ first and second in (16) so

$$\frac{dx}{x} = \frac{du}{1},$$

so

$$u - \ln|x| = c_2.$$

The first and third in (16) integrates to

$$\frac{p}{x^2} = c_3,$$

and the first and forth in (16) integrates to

$$\frac{q}{x^2} = c_4.$$

Thus, the solution to the system (13) is

$$\frac{y}{x} = c_1, \quad u - \ln|x| = c_2, \quad \frac{p}{x^2} = c_3, \quad \frac{q}{x^2} = c_4.$$