Math 3331 – ODEs Sample Test 2 Solutions

1. Let $A = A(t)$ be the amount of salt at any time. Initially the tank contains pure water so $A(0) = 0$. The rate in is $r_i = 5$ gal/min and rate out $r_o = 10$ gal/min meaning the volume in the tank is decreasing so

$$V = V_0 + (r_i - r_o)t = 500 + (5 - 10)t = 500 - 5t$$

The change in salt at any time is given by

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

where c_i and c_o are concentrations in and out. Since we are given that $c_i = 2$ lb/gal and $c_o = A(t)/V(t)$ then we have

$$\begin{aligned} \frac{dA}{dt} &= 2 \cdot 5 - 10 \cdot \frac{A}{500 - 5t} \\ &= 10 - \frac{2A}{100 - t} \end{aligned}$$

This is linear so

$$\frac{dA}{dt} + \frac{2A}{100 - t} = 10$$

The integrating factor is $\mu = \exp\left(\int \frac{2}{100 - t} dt\right) = \exp\left(-2 \ln|100 - t|\right) = 1/(100 - t)^2$ so

$$\frac{d}{dt}\left(\frac{A}{(100 - t)^2}\right) = \frac{10}{(100 - t)^2}$$

Integrating gives

$$\frac{A}{(100 - t)^2} = \frac{10}{(100 - t)} + c$$

The initial condition $A(0) = 0$ gives $c = -1/10$ and finally giving the amount of salt at any time

$$A = 10(100 - t) - \frac{1}{10}(100 - t)^2.$$

When the tank is empty $V = 0$ which happens at $t = 100$ and $A(100) = 0$.

2. Let $P = P(t)$ be the population of rabbits. The differential equation is

$$\frac{P}{dT} = kP(1000 - P)$$

Separating gives

$$\frac{dP}{P(1000 - P)} = kdt$$

or

$$\frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = kdt$$

and multiplying by 1000

$$\left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = 1000kdt$$

We can absorb the 1000 into the k. Integrating gives

$$\ln P + \ln(1000 - P) = kt + \ln c$$

or

$$\frac{P}{1000 - P} = ce^{kt} \tag{1}$$

Using the initial condition gives $P(0) = 100$ gives $c = 1/9$ and further $P(1) = 120$ gives $k = .204794$. Solving (1) for P gives

$$P = \frac{1000e^{.204794t}}{e^{.204794t} + 9}$$

3. Assuming Newton's law of cooling we have

$$\frac{dT}{dt} = k\left(T_\infty - T\right)$$

subject to $T(0) = 160$ and $T(20) = 150$. Here $T_\infty = 70$. Separating the DE gives

$$\frac{dT}{70 - T} = kdt$$

which we write as

$$\frac{dT}{T - 70} = -kdt$$

as T is greater than the room temperature 70. Integrating gives

$$\ln|T - 70| = -kt + \ln c$$

or

$$T = 70 + ce^{-kt}$$

Using $T(0) = 160$ gives $c = 90$ and using $T(20) = 150$ gives $k = .005882$. Thus, the temperature at any time is given by

$$T = 70 + 90e^{-.005882t}.$$

4. Solve the following

$(i) \quad y'' - 5y' + 6y = 0, \quad y(0) = 1, \; y'(0) = 0$

Soln: The CE is $m^2 - 5m + 6 = 0$ so $(m-2)(m-3) = 0$ giving $m = 2, m = 3$. The solution is

$$y = c_1 e^{2x} + c_2 e^{3x}$$

The IC's gives $c_1 + c_2 = 1, 2c_1 + 3c_2 = 0$. Solving gives $c_1 = 3, c_2 = -2$ leading to the solution

$$y = 3e^{2x} - 2e^{3x}$$

$(ii) \quad y'' + 2y' + 10y = 0, \quad y(0) = -1, \; y'(0) = 4$

Soln: The CE is $m^2 + 2m + 10 = 0$ giving $m = -1 \pm 3i$. The solution is

$$y = c_1 e^{-x} \cos 3x + c_2 e^{-x} \sin 3x$$

The IC's gives $c_1 = -1, -c_1 + 3c_2 = 4$. Solving gives $c_1 = -1, c_2 = 1$ leading to the solution

$$y = -e^{-x} \cos 3x + e^{-x} \sin 3x$$

$(iii) \quad 4y'' - 4y' + y = 0, \quad y(0) = 0, \; y'(0) = 1$

Soln: The CE is $4m^2 - 4m + 1 = 0$ so $(2m-1)(2m-1) = 0$ giving $m = 1/2, m = 1/2$. The solution is

$$y = c_1 e^{1/2x} + c_2 x e^{1/2x}$$

The IC's gives $c_1 = 0, c_2 = 1$ leading to the solution

$$y = xe^{1/2x}$$

5. (i) Solve

$$x^2 y'' - xy' + y = 0,$$

given that $y_1 = x$ is one solution.

Soln: Let $y = xu$ so $y' = xu' + u$ and $y'' = xu'' + 2u'$. Substituting and simplifying gives

$$x^3u'' + x^2u' = 0$$

Letting $u' = v$ so $u'' = v'$ gives

$$x^3v' + x^2v = 0.$$

Separating gives

$$\frac{dv}{v} = -\frac{1}{x}\,dx$$

which integrates to

$$v = \frac{1}{x}.$$

Since $u' = v$ this integrates once more giving

$$u = \ln|x|$$

and since $y = xu$ we obtain the second solution $y = x\ln|x|$. Thus the general solution is

$$y = c_1x + c_2x\ln|x|$$

5. (ii) Solve

$$xy'' - (x+1)y' + y = 0,$$

given that $y_1 = e^x$ is one solution.

Soln: Let $y = e^xu$ so $y' = e^xu' + e^xu$ and $y'' = e^xu'' + 2x^xu' + e^xu$. Substituting and simplifying gives

$$xu'' + (x+1)u' = 0$$

Letting $u' = v$ so $u'' = v'$ gives

$$xv' + (x-1)v = 0.$$

Separating gives

$$\frac{dv}{v} = \frac{1-x}{x}\,dx$$

which integrates to

$$v = xe^{-x}.$$

Since $u' = v$ this integrates once more giving

$$u = -(x+1)e^{-x}$$

and since $y = e^xu$ we obtain the second solution $y = -(x+1)$. Thus the general solution is

$$y = c_1e^x + c_2(x+1)$$

noting that we absorbed the -1 into c_2.