Math 6345 - Adv. ODE's

Ex MIT $\qquad \dot{\bar{x}} = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} \bar{x}$

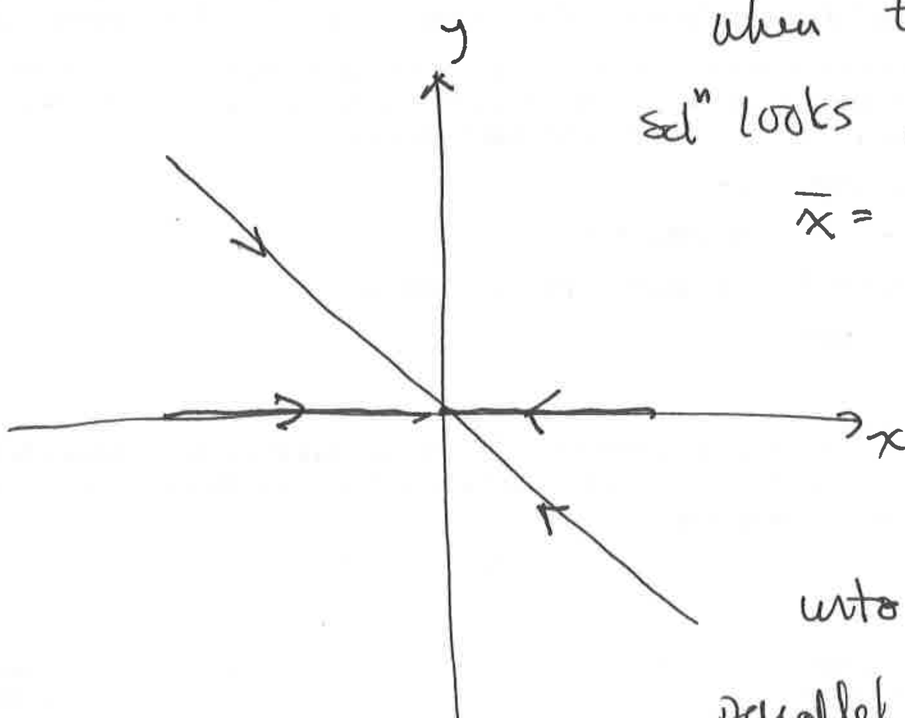
Eigenvalues $\qquad \begin{vmatrix} \lambda+1 & -2 \\ 0 & \lambda+3 \end{vmatrix} = 0 \qquad (\lambda+1)(\lambda+3) = 0$

$\lambda = -1, -3$

Eigenvectors $\qquad \lambda = -1 \qquad\qquad \lambda = -3$

$\begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix} \bar{u} = 0 \qquad \bar{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix} \bar{u} = 0 \qquad \bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

sol^n $\qquad \bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$



when $t \to \infty$

sol^n looks like

$\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$

so all sol^{n}s

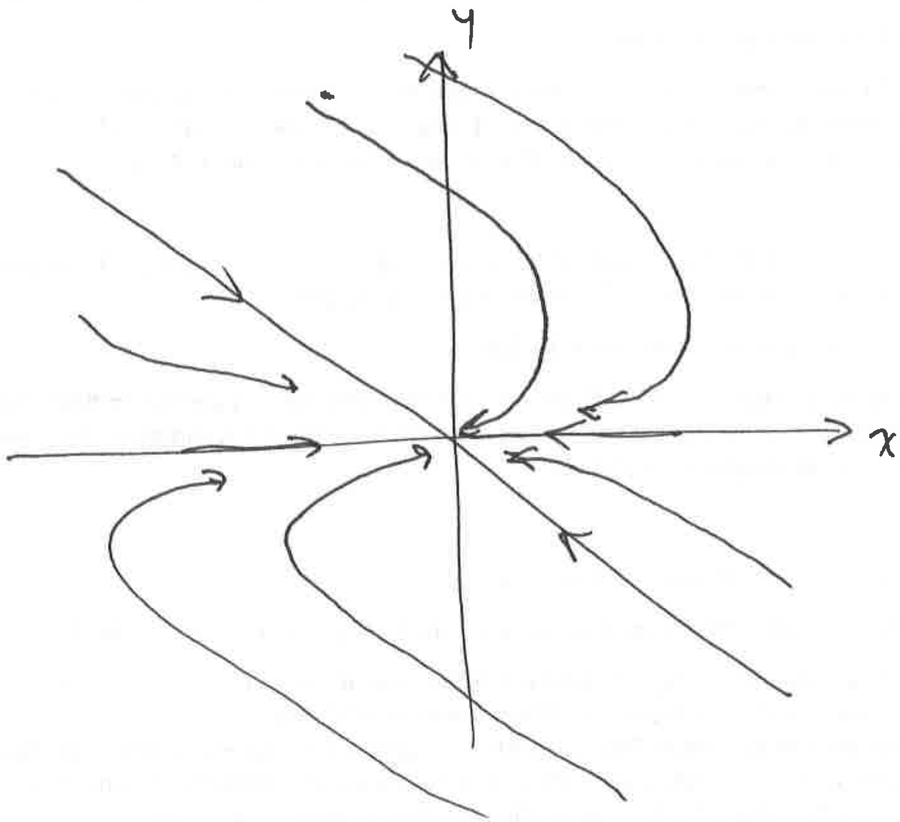
will come

into the origin

parallel to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

when $t \to -\infty$ all sol^n's will look like

$$\bar{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$



so the phase plane looks like ↑

Math 6345 – Adv. ODEs

Ex 1

$$\frac{dx}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x}$$

$$|\lambda \underline{I} - A| = \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0 \Rightarrow \quad \lambda^2 - 4\lambda + 4 - 1 = 0$$

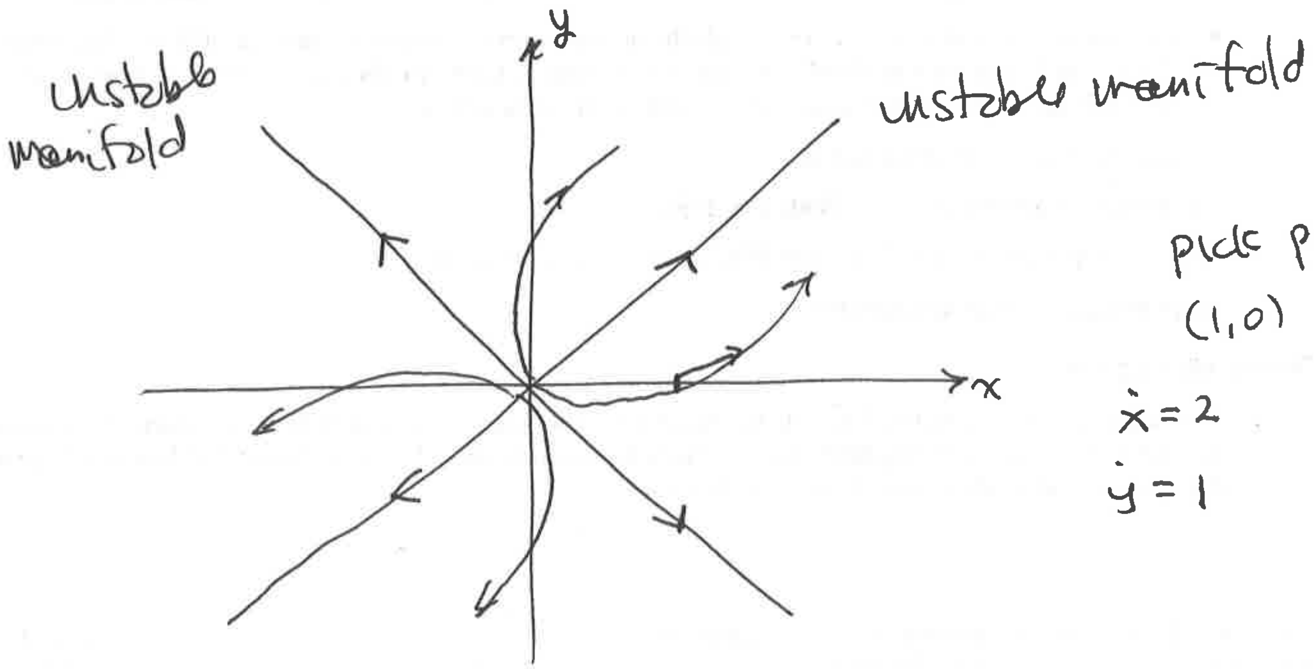
$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1, 3$$

$\lambda = 1$ $\quad \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad u + v = 0 \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 3$ $\quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad u - v = 0 \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Sol^{n} $\quad \bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$



pick point $(1,0)$

$\dot{x} = 2$

$\dot{y} = 1$

Ex2 Real Distinct Opposite Eigenvalue

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix} \bar{x}$$

Eigenvalues $\begin{vmatrix} \lambda+3 & 4 \\ -1 & \lambda-2 \end{vmatrix} = 0$ $\quad (\lambda+3)(\lambda-2)+4=0$

$\lambda^2+\lambda-2=0$

$(\lambda+2)(\lambda-1)=0$

$\lambda = 1, -2$

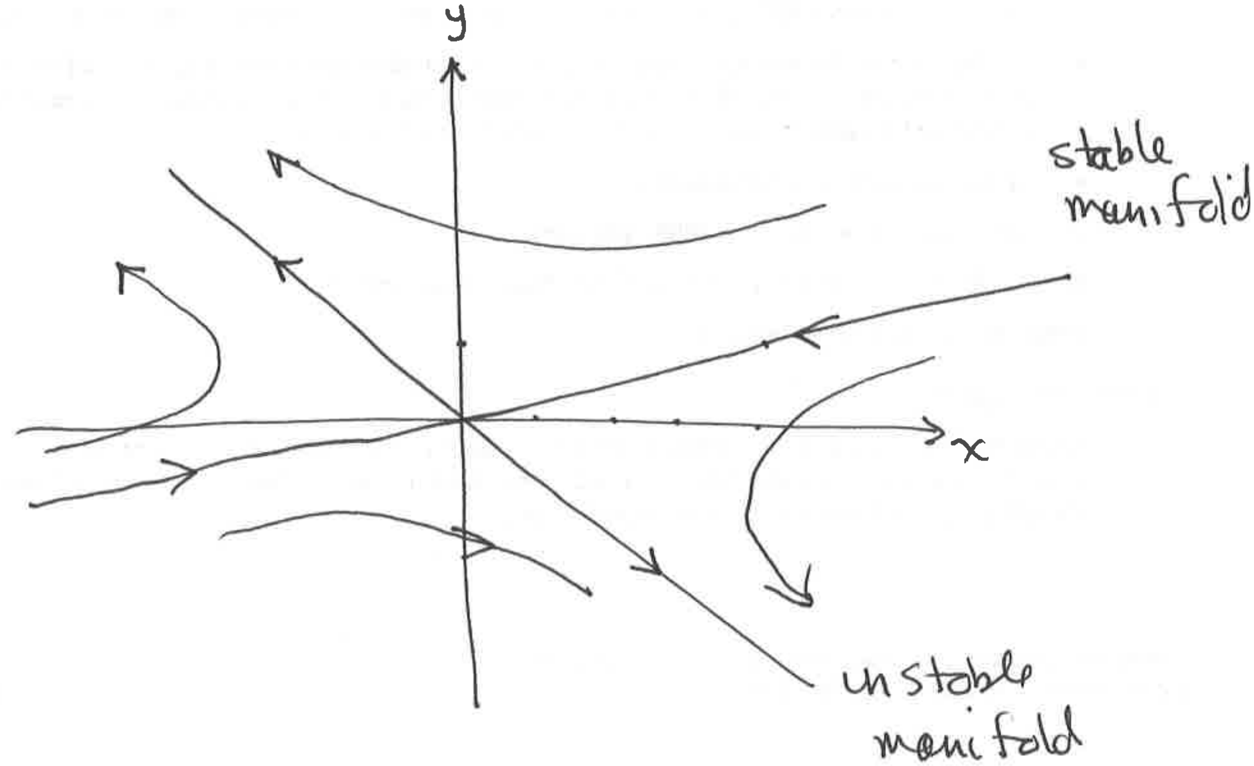
$\lambda = -2$

$\begin{pmatrix} 1 & 4 \\ -1 & -4 \end{pmatrix} \bar{u} = \bar{0} \quad \bar{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Sol^n $\bar{x} = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$

$\lambda = 1$

$\begin{pmatrix} 4 & 4 \\ -1 & -1 \end{pmatrix} \bar{u} = 0 \quad \bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Ex3 A zero Eigenvalue

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \bar{x}$$

so $\begin{vmatrix} \lambda-2 & 1 \\ 2 & \lambda-1 \end{vmatrix} = 0$ $\quad \lambda^2 - 3\lambda + 2 - 2 = 0$

$\lambda(\lambda-3) = 0 \quad \lambda = 0, 3$

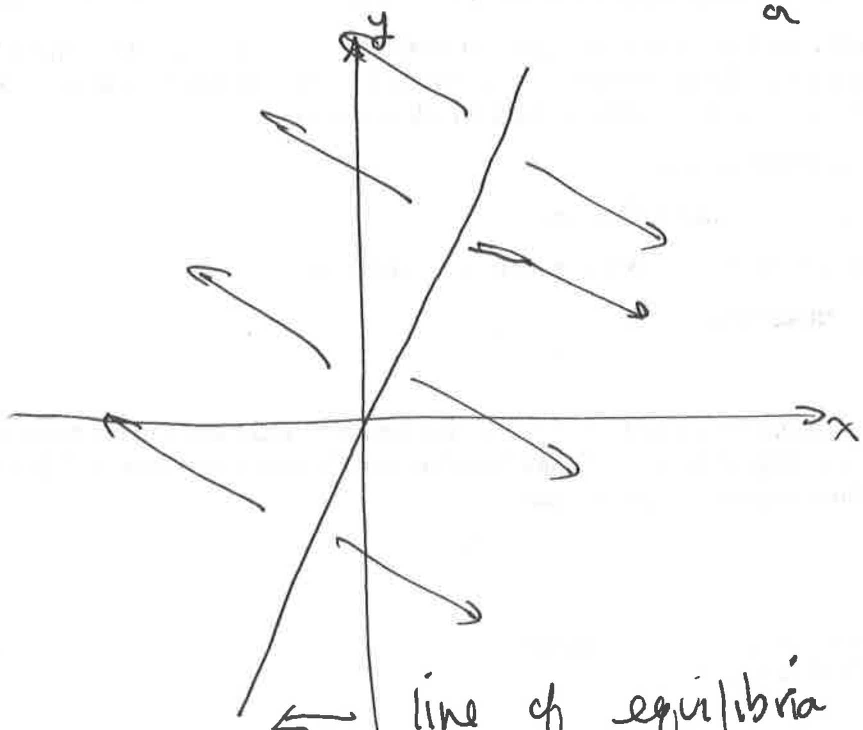
$\lambda = 0 \quad \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \bar{u} = \bar{0} \Rightarrow -2u + v = 0 \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = 3 \quad \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \bar{u} = \bar{0} \Rightarrow u + v = 0 \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

sol^n $\bar{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$ $\quad$ as $t \to -\infty$

$x \to c_1 \quad y \to 2c_1$

or $y = 2x$



Note $\quad \left.\begin{aligned} \frac{dx}{dt} &= 2x - y \\ \frac{dy}{dt} &= -(2x-y) \end{aligned}\right\} \quad \frac{dy}{dx} = -1$ alway

In general

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bar{x} \qquad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

A zero eigen value $\Rightarrow$ $ad - bc = 0$

suppose $a \neq 0$ then $d = \frac{bc}{a}$

so $\frac{dx}{dt} = ax + by \quad \frac{dy}{dt} = cx + dy = cx + \frac{bc}{a}y$

$$= \frac{c}{a}(ax + by)$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{c}{a}$ constant - alway

Also on the line $ax + by = 0$

$\dot{x} = 0, \ \dot{y} = 0$ so a line of equilibria

Repeated Eigenvalues

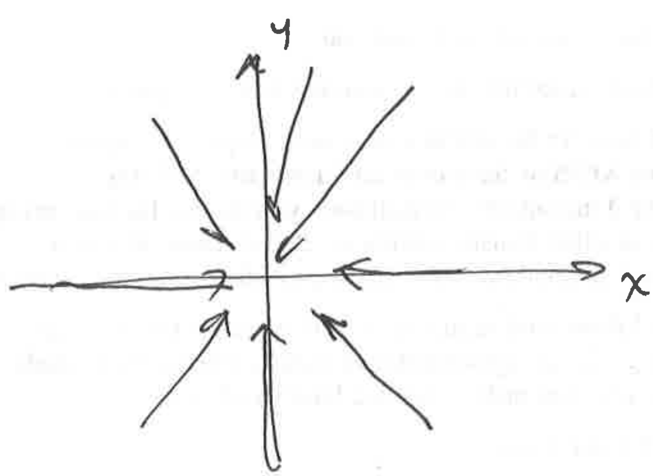
Ex $\dot{\bar{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \bar{x}$

Eigenvalues $\lambda = -1, -1$ sol^n $x = c_1 e^{-t}, y = c_2 e^{-t}$

$\bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$

so 2 separate eigenvectors (not usual)

Trajectories $y = \frac{c_2}{c_1} x$



Ex

$\dot{\bar{x}} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \bar{x}$

Eigenvalues

$$\begin{vmatrix} \lambda+1 & 1 \\ -1 & \lambda+3 \end{vmatrix} = 0 \qquad (\lambda+1)(\lambda+3)+1=0$$

$$\lambda^2+4\lambda+4=0$$

$$\lambda=-2,-2$$

$\lambda=-2$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \bar{u} = \bar{0} \quad \text{so} \quad \bar{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1 sol^n $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

For second sol^n, seek a sol^n of the form

$$\bar{x}_2 = \bar{p}\, t e^{-2t} + \bar{q}\, e^{-2t}$$

$$\dot{\bar{x}} = A\bar{x} \Rightarrow \bar{p}\, e^{-2t} - 2\bar{p}\, t e^{-2t} - 2\bar{q}\, e^{-2t} = A\bar{p}\, t e^{-2t} + A\bar{q}\, e^{-2t}$$

comparing gives $\quad -2\bar{p} = A\bar{p} \Rightarrow (-2I-A)\bar{p} = 0$

$$\bar{p} - 2\bar{q} = A\bar{q} \Rightarrow (-2I-A)\bar{q} = -\bar{p}$$

the first is the original eigenvalue-vector prob

so $\bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ second $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \bar{q} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ so $\bar{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

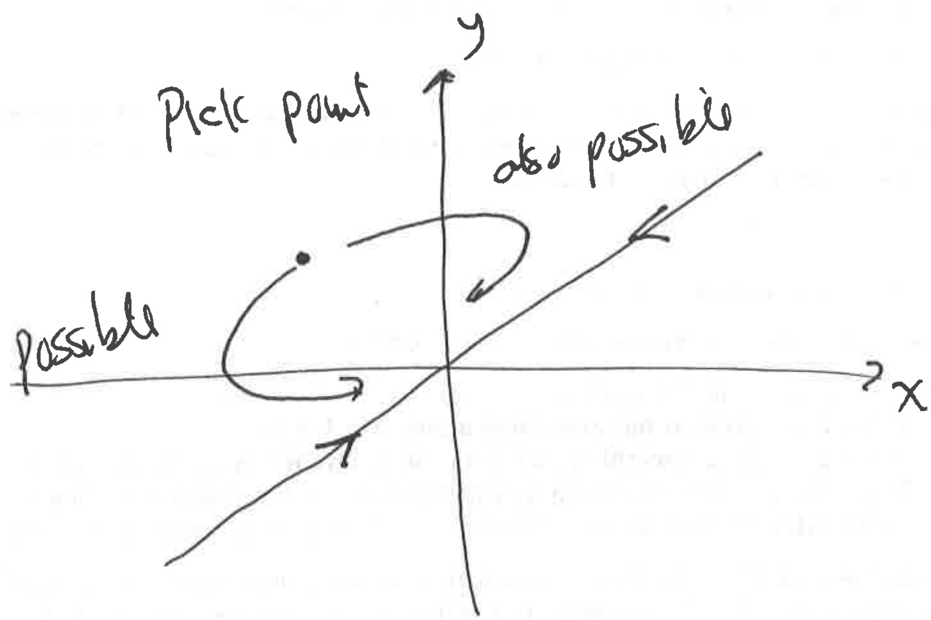
any q satisfying

Sol^n

$$\bar{x} = c_1 \begin{pmatrix}1\\1\end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix}1\\1\end{pmatrix} t e^{-2t} + \begin{pmatrix}1\\0\end{pmatrix} e^{-2t} \right]$$

The most dominant term here is

$$\begin{pmatrix}1\\1\end{pmatrix} t e^{-2t} \quad \text{as } t \to \infty \quad (\text{both } + \text{ \& } - \infty)$$



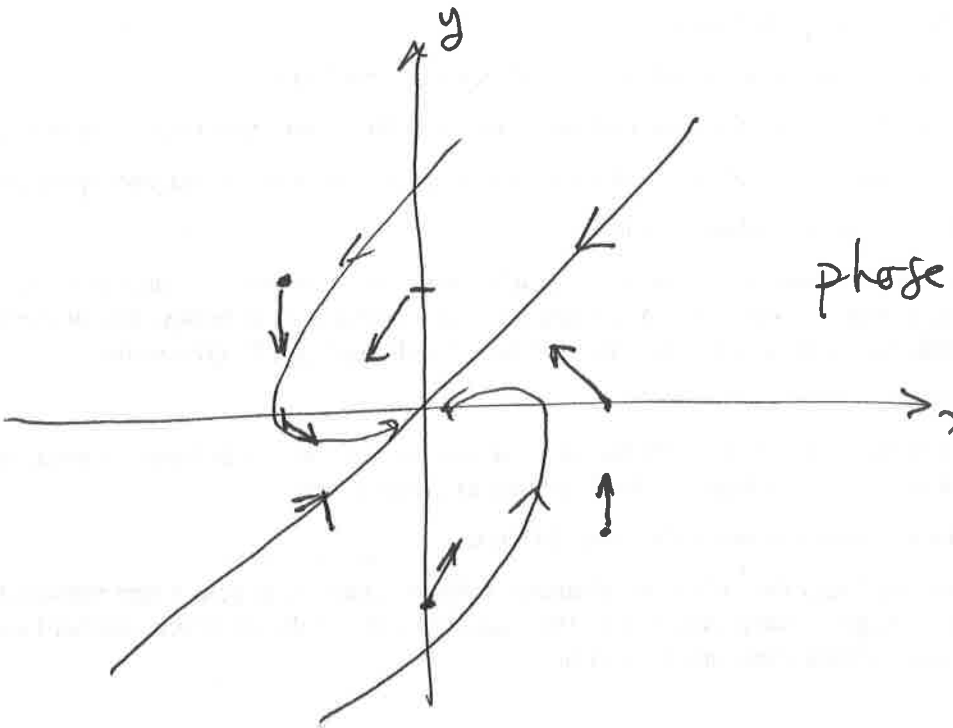
so solution become parallel to $\begin{pmatrix}1\\1\end{pmatrix}$ both forward & backwards in time

so which one? at $(-1,1)$

$$\dot{x} = -(-1) - 1(1) = 0$$

$$\dot{y} = -1 - 3 = -4$$

Pick another pt



phase portrait

at $(0,1)$ $\dot{x} = -1$, $\dot{y} = -3$

at $(-1,0)$ $\dot{x} = 1$, $\dot{y} = -1$

at $(1,0)$ $\dot{x} = +1$, $\dot{y} = +1$

at $(1,-1)$ $\dot{x} = 0$, $\dot{y} = 4$

at $(0,-1)$ $\dot{x} = 1$, $\dot{y} = 3$