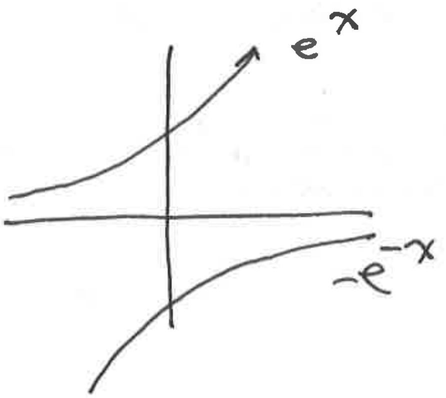
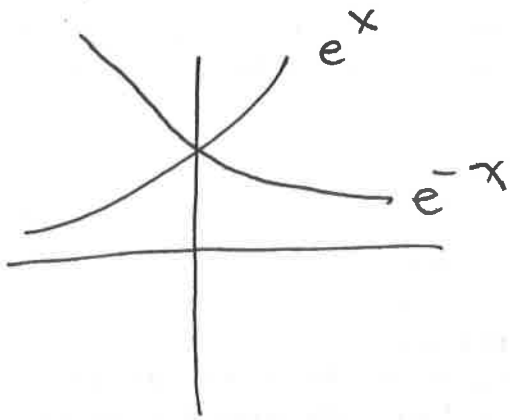
Math 4315 - PDE's

Two New Functions

Hyperbolic Sine & Cosine



$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$



$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

The reason for the "sine" & "cosine" is that they behave (have similar properties" to the trig functions

Ex $\quad \sinh(2x) = \dfrac{e^{2x} - e^{-2x}}{2} = 2\left(\dfrac{e^x - e^{-x}}{2}\right) \cdot \dfrac{e^x + e^{-x}}{2} = 2\sinh(x)\cosh(x)$

$\quad \cosh(2x) = \dfrac{e^{2x} + e^{-2x}}{2} = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - 1 = \cosh^2(x) - 1$

$\cosh^2 x - \sinh^2 x = 1$ <u>note</u>: sign change

Laplace's Eq^n

Solve $u_{xx} + u_{yy} = 0$ $\quad 0 < x < 1$

$\qquad 0 < y < 1$

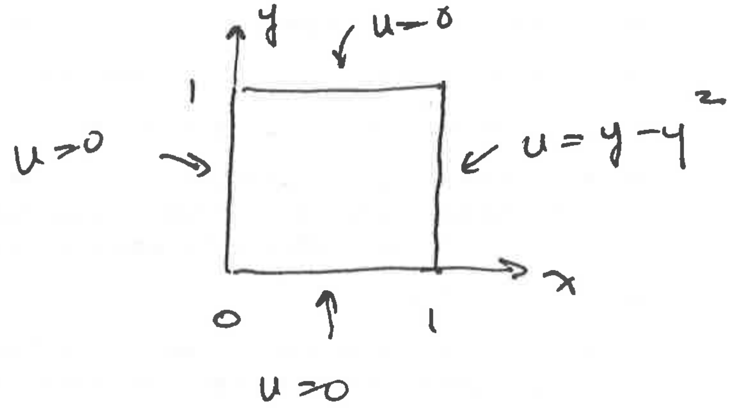
subject to

$u(x,0) = 0$

$u(x,1) = 0$

$u(0,y) = 0$

$u(1,y) = y - y^2$



Again we will assume separable sol^n

$$u = X(x)\,Y(y)$$

sub into PDE

$$X''Y + XY'' = 0$$

& sep $\quad \dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$ $\quad$ each is constant so

$$\Rightarrow \quad \frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda \quad \text{(they add to zero)}$$

Now the B.C.

$u(x,0) = 0 \Rightarrow X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0$

$u(x,1) = 0 \Rightarrow X(x)\,Y(1) = 0 \Rightarrow Y(1) = 0$

$u(0,y) = 0 \Rightarrow X(0)\,Y(y) = 0 \Rightarrow X(0) = 0$

$u(1,y) = y - y^2$ - reserved for the end

so to solve

$$\frac{Y''}{Y} = -\lambda \qquad Y(0) = 0 \quad Y(1) = 0$$

we need $\lambda > 0$ so let $\lambda = \omega^2$

$$\Rightarrow \quad Y'' + \omega^2 Y = 0$$

sol^n $\quad Y = c_1 \sin \omega y + c_2 \cos \omega y$

like we saw earlier $\quad Y(0) = 0 \Rightarrow c_2 = 0$

$\qquad Y(1) = 0 \Rightarrow \omega = n\pi$

so $\quad Y = c_1 \sin n\pi y$

so now we consider the X eq^n

$$\frac{X''}{X} = \lambda = \omega^2$$

so $X'' - \omega^2 X = 0$

$$\Rightarrow \quad X = c_3 e^{\omega x} + c_4 e^{-\omega x}$$

(we'll bring the hyperbolic soon)

$X(0) = 0 \Rightarrow \quad c_3 + c_4 = 0 \qquad c_2 = -c_1$

$$X = c_3 e^{\omega x} - c_3 e^{-\omega x}$$

$$= c_3 \left(e^{\omega x} - e^{-\omega x}\right)$$

$$= c_3 \sinh \omega x$$

the 2 came in via the c_3

$$= c_3 \sinh n\pi x$$

so $u = XY$

$$= c_1 c_3 \sinh n\pi x \sin n\pi y$$

$$u = \sum_{n=1}^{\infty} c_n \sinh n\pi x \sin n\pi y$$

Now the remaining B.C.

$$u(1,y) = y - y^2$$

so $$y - y^2 = \sum_{n=1}^{\infty} C_n \sinh n\pi \sin n\pi y$$

let $$C_n \sinh n\pi = b_n$$

so $$y - y^2 = \sum_{n=1}^{\infty} b_n \sin n\pi y$$

Fourier sine series

$$b_n = \frac{2}{1}\int_0^1 (y-y^2)\sin n\pi y$$

$$= \frac{4(1-\cos n\pi)}{n^3\pi^3} \qquad \text{so} \qquad C_n = \frac{b_n}{\sinh n\pi}$$

so $$u = \sum \frac{4(1-\cos n\pi)}{n^3\pi^3} \frac{\sinh n\pi x}{\sinh n\pi} \cdot \sin n\pi y$$

soln