Math 4315 – PDE's

Solve

$$u_t = u_{xx} + au, \quad a \in \mathbb{R}$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

$$u(x,0) = x - x^2$$

we could do separation of variables but instead we will try and transform it to a problem we already know how to solve.

Try $\quad u(x,t) = A(t)\, v(x,t)$

Now BC.

$$\left.\begin{array}{l} u(0,t) = 0 \Rightarrow v(0,t) = 0 \\ u(1,t) = 0 \Rightarrow v(1,t) = 0 \end{array}\right\} \text{same}$$

If. $u(x,0) = A(0) v(x,0)$

so it would be nice if $A(0) = 1$

Now the PDE

$$u_t = A(t) v_t + A'(t) v$$

$$u_{xx} = A(t) v_{xx}$$

sub

$$A(t) v_t + A'(t) v = A(t) v_{xx} + a A(t) v$$

want those to cancel

so $A'(t) = a A(t)$

$$\Rightarrow \quad \frac{dA}{A} = a\,dt \qquad \ln A = at + \ln c$$

$$A(t) = c e^{at}$$

$A(0) = 1 \Rightarrow c = 1$ so $A(t) = e^{at}$

So $u(x,t) = e^{at} v(x,t)$

& $u(x,0) = x - x^2 \Rightarrow v(x,0) = x - x^2$

so we solve:

$$v_t = v_{xx} \quad 0 < x < 1$$

$$v(0,t) = 0 \quad v(1,t) = 0$$

$$v(x,0) = x - x^2$$

Already done

$$v = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin n\pi x$$

$$b_n = \frac{2}{1} \int_0^1 (x - x^2) \sin n\pi x \, dx$$

then $u = e^{at} v$

Solve $\quad u_t = u_{xx} + b u_x$

$u(0,t) = 0 \quad u(1,t) = 0, \quad u(x,0) = x - x^2$

Try $\quad u = A(t) V$

sub

$$A(t) V_t + \underline{A'(t)} V = A V_{xx} + \underline{A b} V_x$$

not the same

modify $\quad u = A(t,x) V$

$u_t = A V_t + A_t V$

$u_x = A V_x + A_x V, \quad u_{xx} = A V_{xx} + 2 A_x V_x + A_{xx} V$

sub

$$A V_t + A_t V = A V_{xx} + 2 A_x V_x + A_{xx} V$$
$$+ b (A V_x + A_x V)$$

choose $A_t = A_{xx} + bA_x$ (v)

$2A_x + bA = 0$ (v_x)

2-eq^n for A

$2A_x + bA = 0$ $\qquad \frac{dA}{A} = -\frac{b}{2}dx$

$\ln A = -\frac{b}{2}x + \ln k(t)$ $\qquad A = k(t)\, e^{-b/2 x}$

Now sub into next eq^n

$k'(t)\cancel{e^{-\frac{b}{2}x}} = k\left(-\frac{b}{2}\right)^2 \cancel{e^{-b/2x}} + b\left(-\frac{b}{2}\right)k\, \cancel{e^{-b/2x}}$

$k' = +\frac{b^2}{4}k - \frac{b^2}{2}k = -\frac{1}{4}b^2 k$ sep

$k(t) = k_0\, e^{-\frac{b^2}{4}t}$

so $u = k_0 e^{-\frac{b}{2}x - \frac{b^2}{4}t} v(x,t)$

so $u_t = u_{xx} + b u_x \Rightarrow v_t = v_{xx}$

Next B.C.

$u(0,t) = 0, \ u(1,t) = 0 \Rightarrow v(0,t) = v(1,t) = 0$

I.C.

$u(x,0) = x - x^2$

$\Rightarrow x - x^2 = k_0 e^{-b/2 x} v(x,t)$ (set $k_0 = 1$)

so $v(x,t) = (x - x^2) e^{b/2 x}$ new BC.

once we solve for v

$u = e^{-b/2 x - b^2/4 t} v$