Edge Augmentation with Controllability Constraints in Directed Laplacian Networks

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Controllability and Robustness in Networks

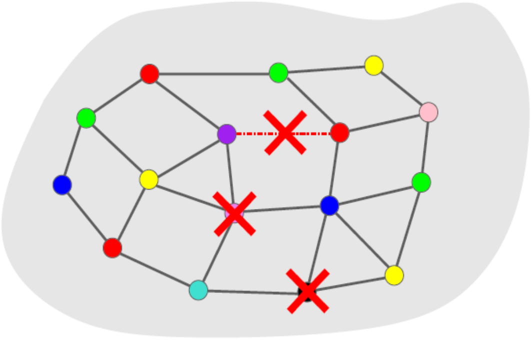
Controllability and robustness are crucial attributes of a networked dynamical system.

Network Controllability

A leader robot begins driving in a predefined path around the room.

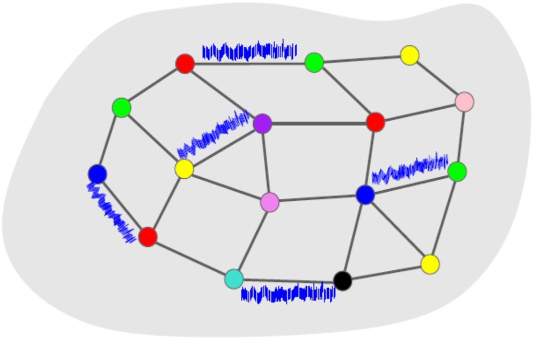
How can we drive a network of agents from some initial state to a final state by controlling only a small subset of agents, referred to as *leaders*?

Network Robustness



How can we minimize the effect of node/edge removals on the overall network structure?

Structural aspect

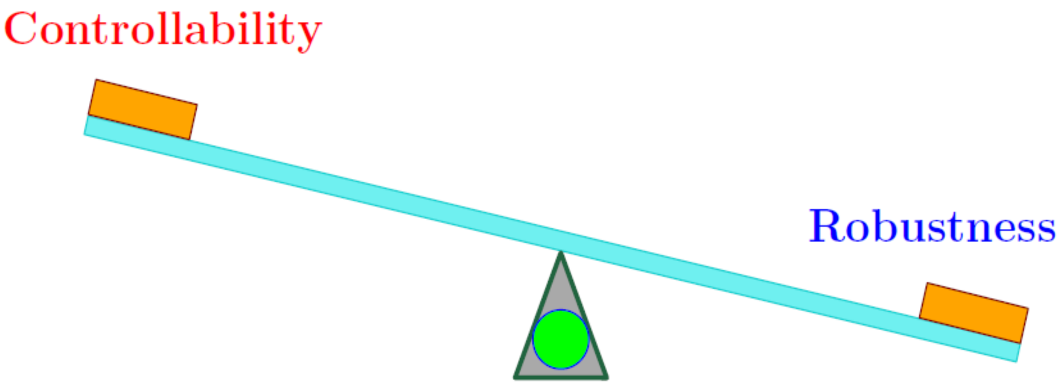


How can we minimize the effect of noisy information on the network's overall performance

Functional aspect

Controllability and Robustness in Networks

Controllability and robustness properties in networks are *conflicting* at times[1,2].



How can we improve one property (for instance, by modifying the network graph) without deteriorating the other property?

[1] F. Pasqualetti, C. Favaretto, S. Zhao, and S. Zampieri, “Fragility and controllability tradeoff in complex networks,” *ACC* 2018.

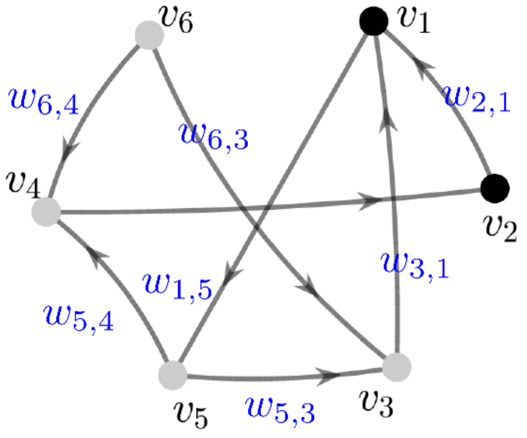
[2] W. Abbas, M. Shabbir, M. Yazicioğlu and A. Akber, “Trade-off Between Controllability and Robustness in Diffusively Coupled Networks,” *IEEE TCNS* 2020.

System and Network Controllability

Network: A network of agents modeled by a *directed graph* $G(V,E)$.

Network dynamics: $\dot{x} = -L_w x + Bu$ (*Weighted Laplacian* dynamics)

$$L_w = \begin{bmatrix} w_{1,5} & 0 & 0 & 0 & -w_{1,5} & 0 \\ -w_{2,1} & w_{2,1} & 0 & 0 & 0 & 0 \\ -w_{3,1} & 0 & w_{3,1} & 0 & 0 & 0 \\ 0 & -w_{4,2} & 0 & w_{4,2} & 0 & 0 \\ 0 & 0 & -w_{5,3} & -w_{5,4} & (w_{5,3}+w_{5,4}) & 0 \\ 0 & 0 & -w_{6,3} & -w_{6,4} & 0 & (w_{6,3}+w_{6,4}) \end{bmatrix}$$



$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(*Input matrix*)

$m \times n$

No. of leaders.

Network (strong structural) Controllability

Controllability matrix:

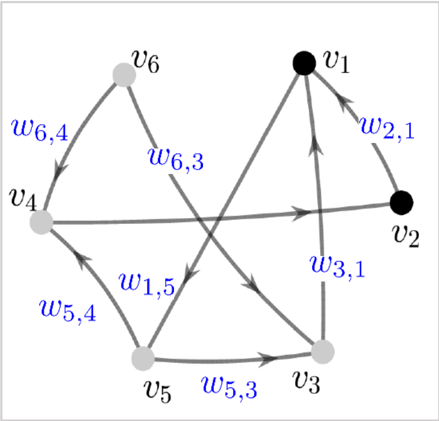
$$\Gamma(L_w, B) = \left[\ B \quad -L_w B \quad (-L_w)^2 B \quad \cdots \quad (-L_w)^{n-1} B\ \right]$$

Controllable subspace: **Rank (Γ)**

Controllability measure: **Rank (Γ)**

input matrix B

structure of graph
weights of edges L_w



Often weights are unknown due to system uncertainties. So, we want the controllability notion to be independent of edge weights.

$$\mathcal{L} = \begin{bmatrix} \times & 0 & 0 & 0 & \times & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ \times & 0 & \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & 0 & \times \end{bmatrix}$$

Strong **S**tructural **C**ontrollability

$$\Gamma = \left[\ B \quad -L_w B \quad \cdots \quad (-L_w)^{n-1} B\ \right]$$

$$\min_{\mathbf{w}} \ \mathrm{Rank}(\mathbf{\Gamma})$$

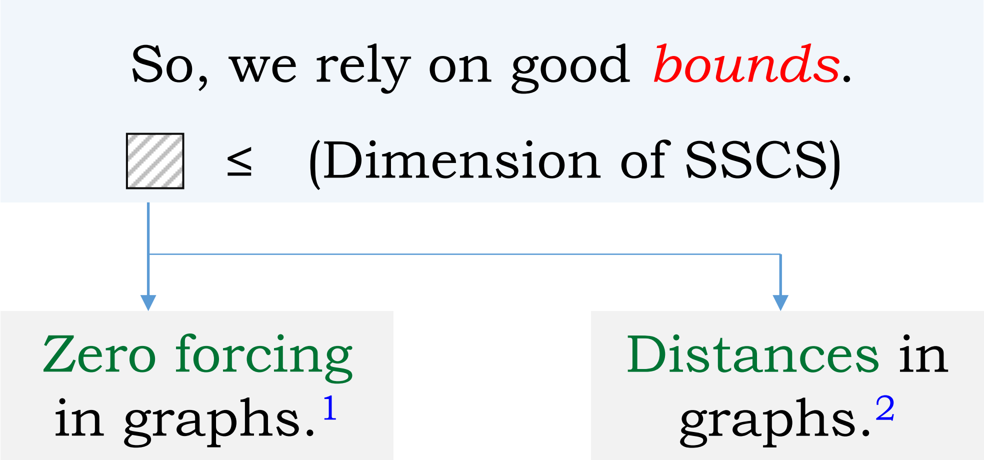
Dimension of SSCS

(measure of SSC)

Edge Augmentation while Preserving Strong Controllability

How can we maximally add edges in a network to improve robustness while preserving its SSC?

Computing the dimension of SSCS is challenging.



So, we rely on good *bounds*.

▨ ≤ (Dimension of SSCS)

Zero forcing in graphs.[1]

Distances in graphs.[2]

Maximally add edges in a network while preserving a lower bound on the dimension of SSC?

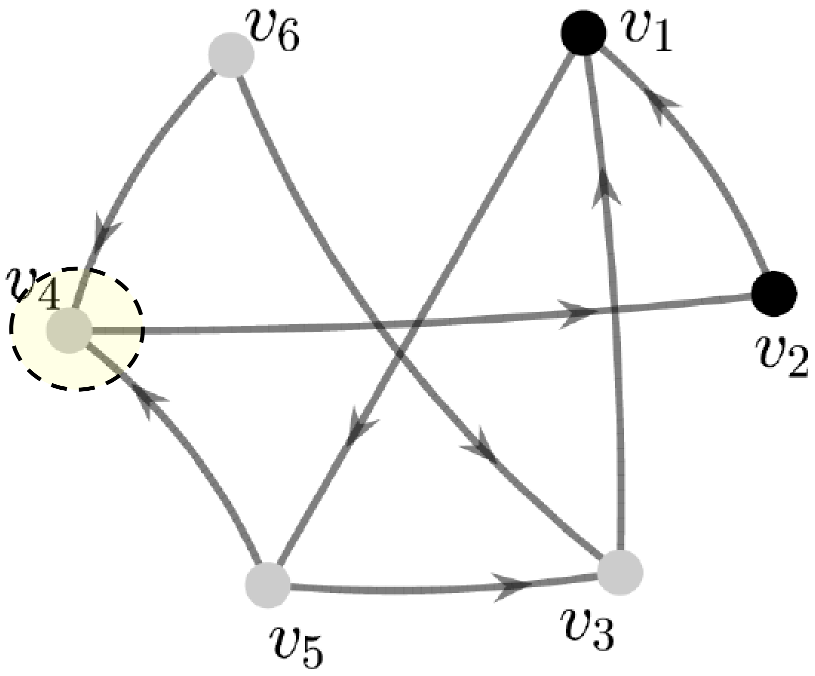
[1] N. Monshizadeh, S. Zhang, and K. Camlibel, “Zero forcing sets and controllability of dynamical systems defined on graphs,” IEEE TAC 2014.

[2] A. Y. Yazıcıoglu, W. Abbas, and M. Egerstedt, “Graph distances and controllability of networks,” IEEE TAC 2016.

Zero Forcing Bound

Zero Forcing Process:

*If a **black** node has v exactly one **white** in-neighbor u, then change the color of u to black. (v **infects** u)*



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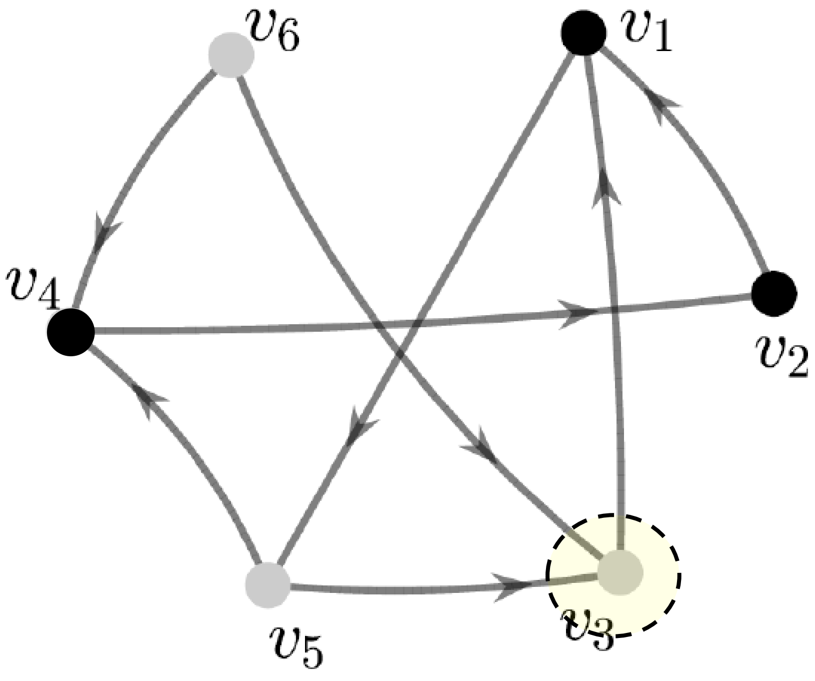

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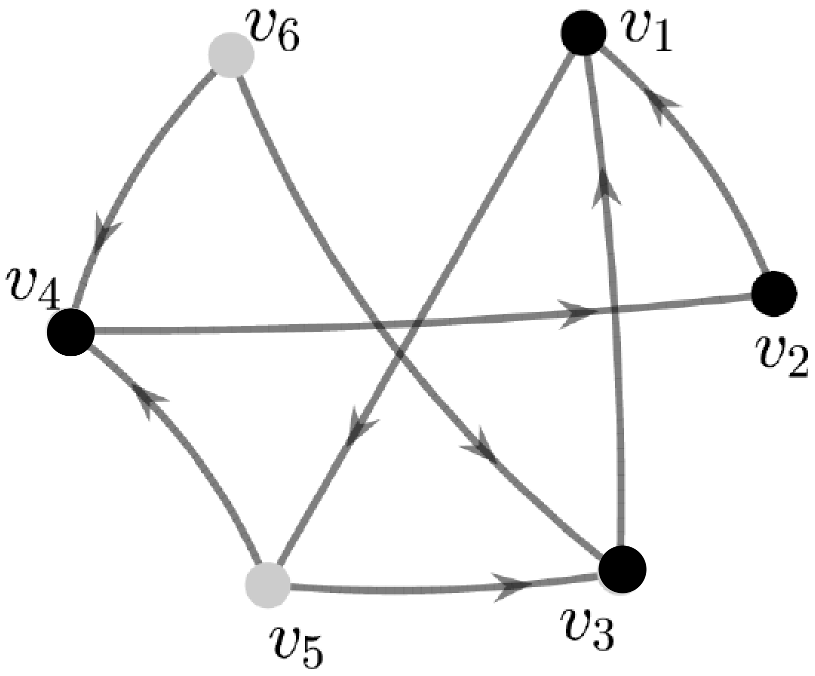
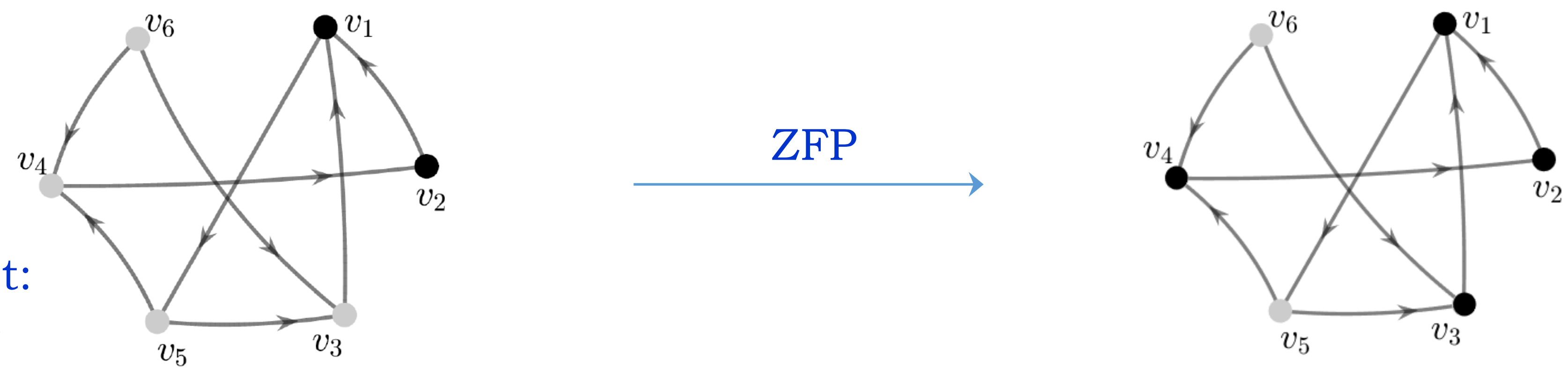


Zero Forcing Set (ZFS)

Derived Set: Set of black nodes at the end of the zero forcing process.

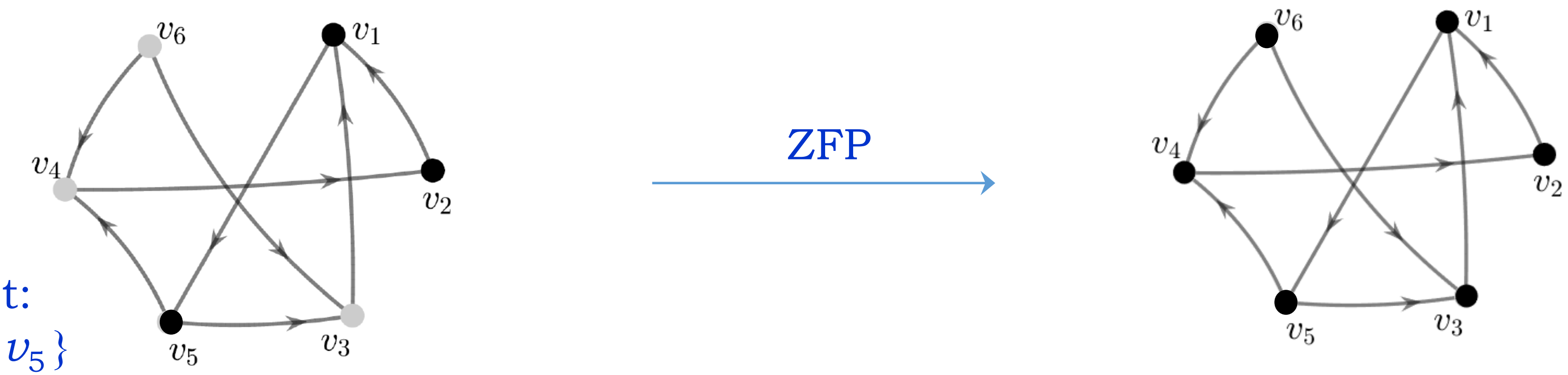
Input set: $\{v_1, v_2\}$



Derived set: $\{v_1, v_2, v_3, v_4\}$

Zero Forcing Set: Initial set of black nodes for which the derived set consists of all nodes in the graph.

ZF set: $\{v_1, v_2, v_5\}$



Derived set: $\{v_1, v_2, \dots, v_6\}$

ZFS and Strong Structural Controllability

Theorem[1]

V_ℓ : Set of leaders (input nodes)

$$\zeta(G, V_\ell) \quad \le \quad \gamma(G, V_\ell)$$

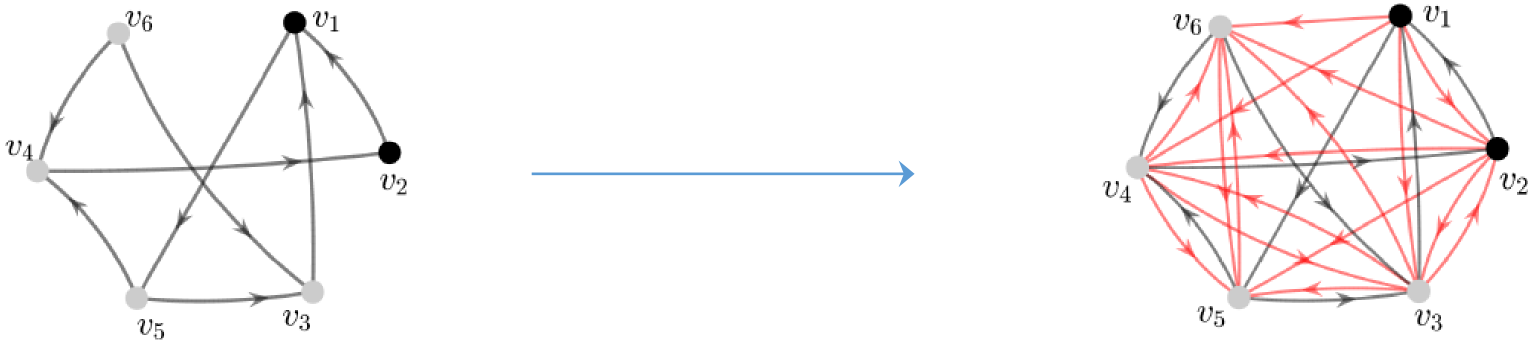
size of the derived set

dimension of SSC

Problem (edge augmentation):

Add maximum edges in G while preserving the size of the derived set.

$$\begin{array}{c} G(V,E) \\ V_\ell \subseteq V \end{array} \quad \xrightarrow{\text{add edges}} \quad \begin{array}{c} G'(V,E') \\ V_\ell\ ,\quad E \subseteq E' \end{array}$$



[1] N. Monshizadeh, K. Camlibel, and H. Trentelman, “Strong targeted controllability of dynamical networks,” IEEE CDC 2015.

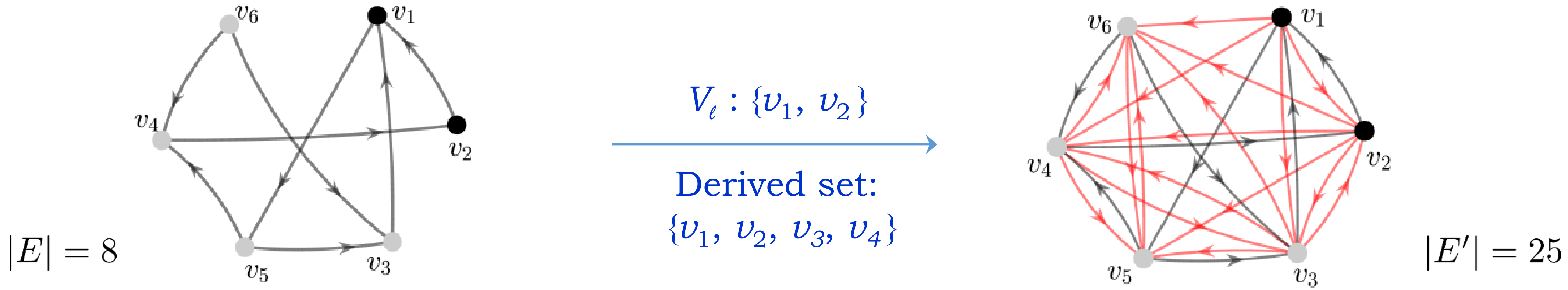
Edge Augmentation using ZFS

Theorem: Given $G(V,E)$ and leader set V_ℓ, Algorithm 1 *optimally* solves the edge augmentation problem and returns $G'(V,E')$, such that

1. Derived set of $G(V,E)$ = Derived set of $G'(V,E')$
2. $|E'|$ is *maximum* while preserving the size of the derived set, and

$$|E'| \;=\; \frac{|\Delta|(|\Delta|+1)}{2} - \frac{m(m+1)}{2} + (m+n-|\Delta|)n - n,$$

where $|V| \;=\; n, \quad |V_\ell| = m, \quad \Delta = \text{derived set}.$



$|E| = 8$

$V_\ell : \{v_1,\, v_2\}$

Derived set: $\{v_1,\, v_2,\, v_3,\, v_4\}$

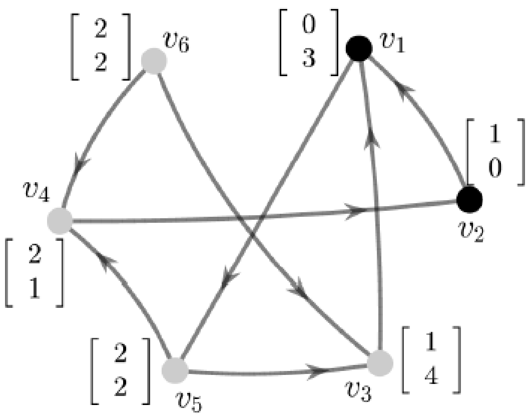
$|E'| = 25$

Distance-based Bound on SSC

Distance-to-leader (DL) vectors

$V_\ell = \{\ell_1, \cdots, \ell_m\}$ (Set of leaders)

$$D_i = \begin{bmatrix} d(v_i, \ell_1) \\ d(v_i, \ell_2) \\ \vdots \\ d(v_i, \ell_m) \end{bmatrix}$$ (DL vector of v_i)


$\begin{bmatrix}2\\2\end{bmatrix}$ v_6
$\begin{bmatrix}0\\3\end{bmatrix}$ v_1
$\begin{bmatrix}1\\0\end{bmatrix}$
v_4
v_2
$\begin{bmatrix}2\\1\end{bmatrix}$
$\begin{bmatrix}2\\2\end{bmatrix}$
$\begin{bmatrix}1\\4\end{bmatrix}$
v_5
v_3

$V_\ell = \{v_1, v_2\}$

Pseudo-monotonically (PMI) increasing sequence

A sequence $\mathcal{D}$ of DL vectors is PMI if for evey i^{th} vector in the sequence, denoted by $\mathcal{D}_i$, there is some $\pi(i) \in \{1, \cdots, m\}$, s.t.

$$[\mathcal{D}_i]_{\pi(i)} < [\mathcal{D}_j]_{\pi(i)}, \quad \forall j > i.$$

$$\mathcal{D} = \left[\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{2} \\ 2 \end{bmatrix} \right].$$

We are interested in the *length* of the PMI sequence.

↓

$\delta(G, V_\ell)$

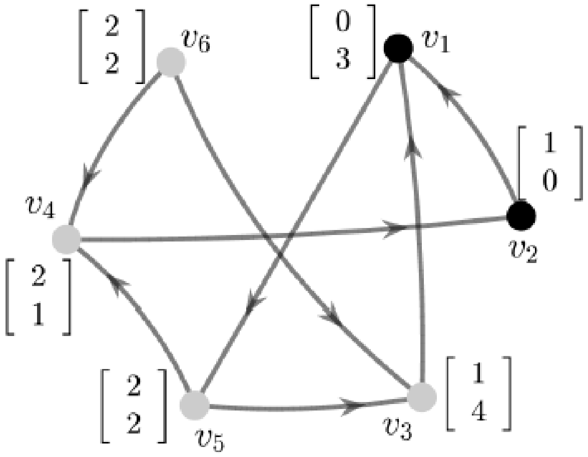
Distance-based Bound on SSC

Theorem[2] V_ℓ : Set of leaders (input nodes)

$$\delta(G, V_\ell) \quad \leq \quad \gamma(G, V_\ell)$$

length of the PMI sequence

dimension of SSC


v_6
$\begin{bmatrix}2\\2\end{bmatrix}$
v_1
$\begin{bmatrix}0\\3\end{bmatrix}$
v_2
$\begin{bmatrix}1\\0\end{bmatrix}$
v_4
$\begin{bmatrix}2\\1\end{bmatrix}$
v_5
$\begin{bmatrix}2\\2\end{bmatrix}$
v_3
$\begin{bmatrix}1\\4\end{bmatrix}$

$V_\ell = \{v_1, v_2\}$

$$\mathcal{D} = \left[\begin{bmatrix}\mathbf{0}\\\mathbf{3}\end{bmatrix}, \begin{bmatrix}1\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\4\end{bmatrix}, \begin{bmatrix}2\\\mathbf{1}\end{bmatrix}, \begin{bmatrix}\mathbf{2}\\2\end{bmatrix} \right].$$

The dimension of SSCS is at least 5 with v_1 and v_2 as leaders.

[2] A. Y. Yazıcıoglu, W. Abbas, and M. Egerstedt, “Graph distances and controllability of networks,” IEEE TAC 2016.

Distance-based Edge Augmentation

Approach:

$G(V, E)$

$V_\ell \subseteq V$

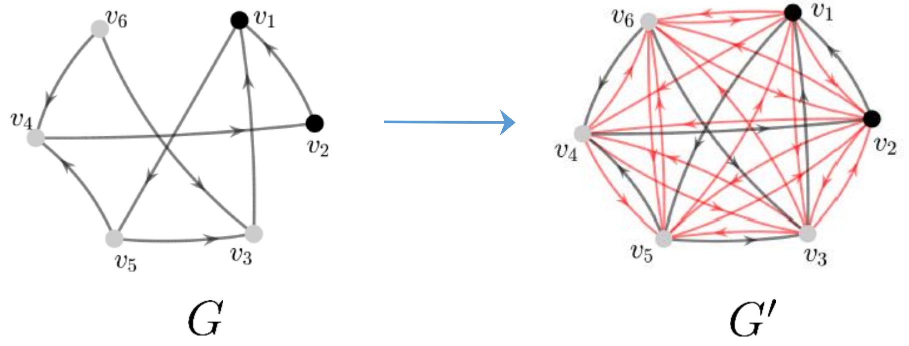
add edges

$G'(V, E')$

$V_\ell\ ,\ \ E \subseteq E'$

$\mathcal{D}$: PMI sequence

$\mathcal{D}$: (same) PMI sequence



Add edges while preserving distances between leaders and *'some'* other nodes.

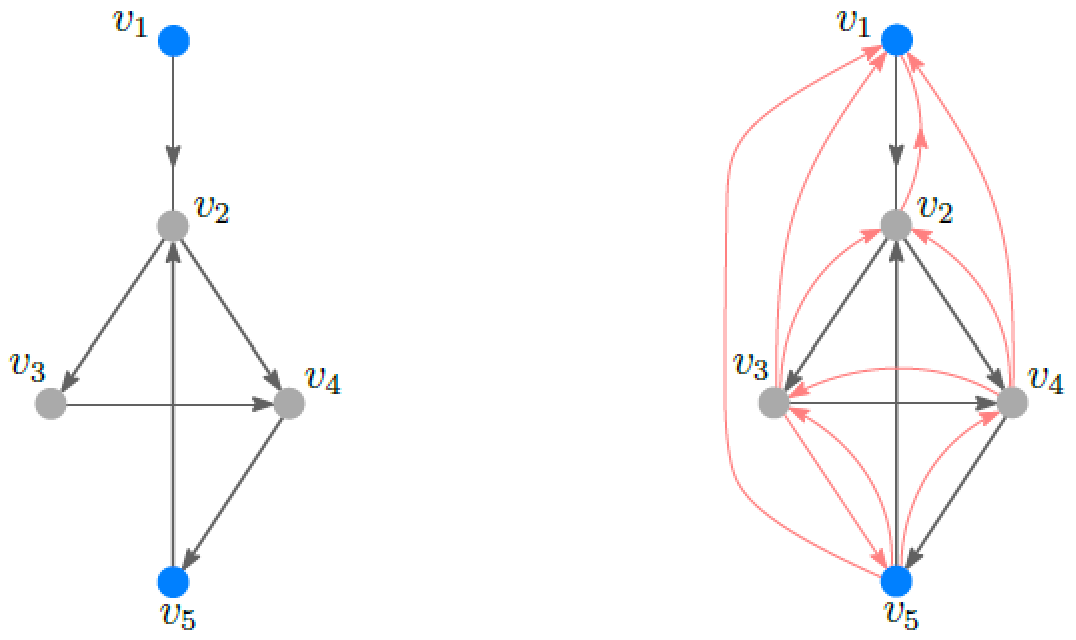
Basic problem:

Given a node pair (a, b), add maximum edges while preserving a distance between them

Distance Preserving Edge Augmentation (DPEA)

DPEA: Given directed G = (V, E), and two nodes a, $b \in$ V such that $d_G(a, b) = k$.

Add maximum no. of edges in G while preserving the distance between a and b.



(a) G

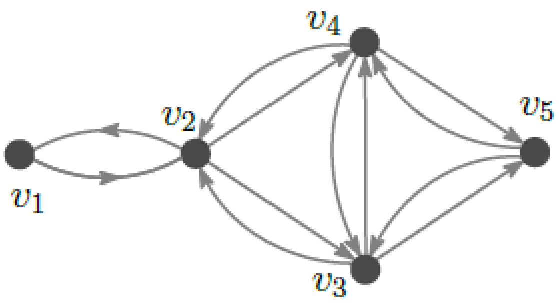
(b) G'

Distance Preserving Edge Augmentation (DPEA)

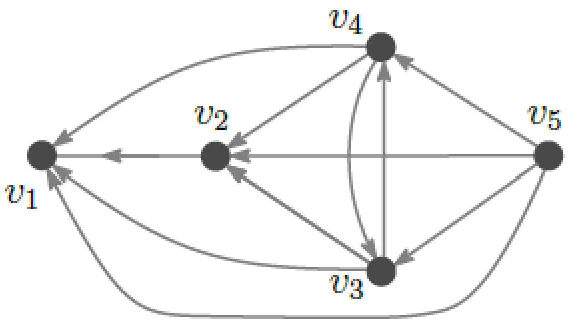
Theorem: For a given node pair (a, b) in $G(V,E)$, an **optimal** solution of the DPEA problem is a union of two graphs called as *clique chain* and *modified clique chain*.

Example: $(a, b) = (v_1, v_5)$

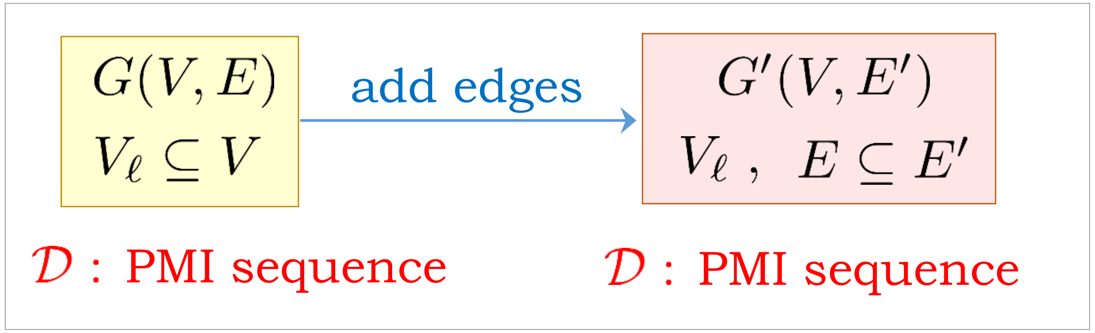
Clique Chain:



Modified Clique Chain:



Distance-based Edge Augmentation



$G(V,E)$

$V_\ell \subseteq V$

add edges

$G'(V,E')$

$V_\ell\ ,\quad E \subseteq E'$

$\mathcal{D}$: PMI sequence

$\mathcal{D}$: PMI sequence

Add edges while preserving distances between leaders and *'some'* nodes.

Obtain all missing edges ***E'*** .

Randomly select a missing edge ***e*** ∊ ***E'***.

If adding ***e*** does not change distances between desired node pairs, then keep it. Otherwise, discard it.

Repeat until no more missing edge is left

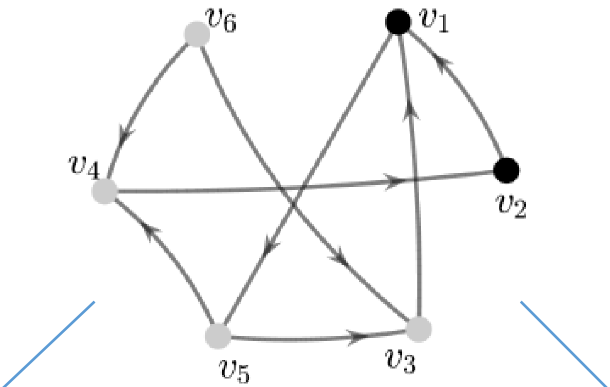
Theorem: Randomized algorithm returns an *α-approximate* solution of the distance-based edge augmentation problem with a *"certain probability"*.

Distance-based vs Zero Forcing Edge Augmentation

Remark: The distance-based edge augmentation often gives better results compared to the zero-forcing-based augmentation, especially when the graph is not SSC

$V_\ell = \{v_1, v_2\}$

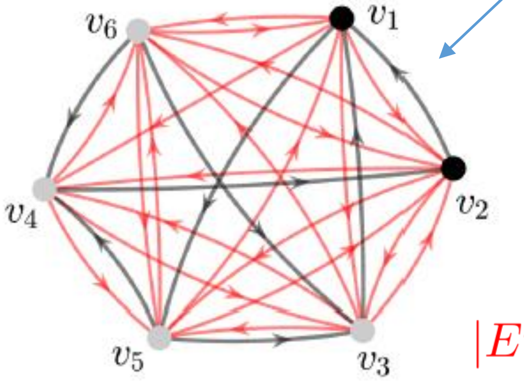
$|E| = 8$



$G = (V, E)$

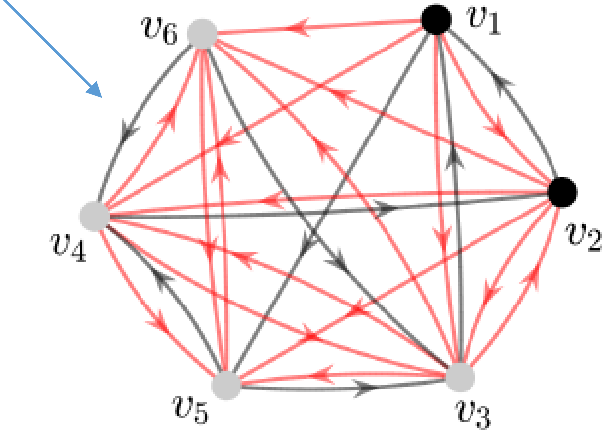
Lower bound on the dim. of SSCS is 4

distance-based



$|E'| = 29$

ZF-based



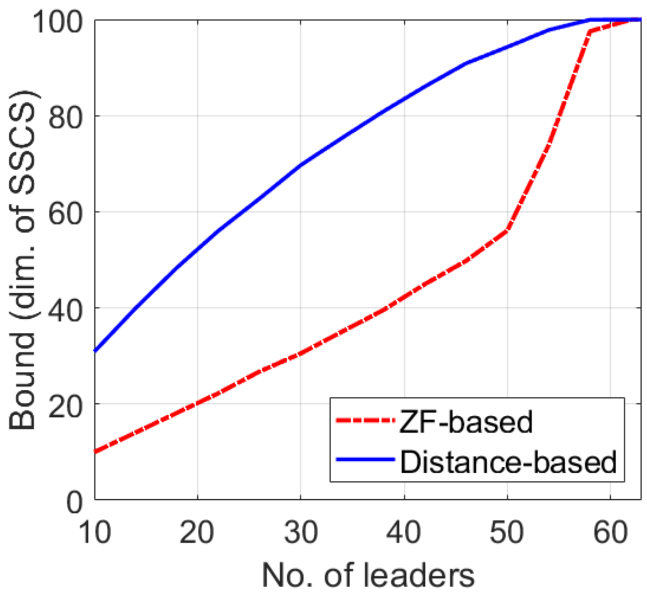
$|E'| = 25$

Numerical Illustrations

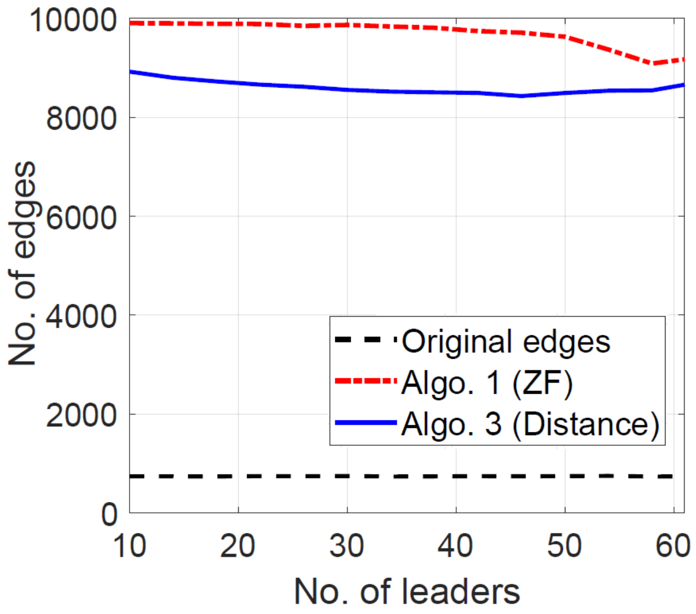
Random Graphs

N = 100, p = 0.075

(Each point is an average of 30 randomly generated instances.)



Lower bound on the dimension of SSC as a function of no. of leaders.



*Number of edges added by ZF-based and distance-based augmentation algorithms while preserving **their respective** bounds.*

Numerical Illustrations

Random Graphs

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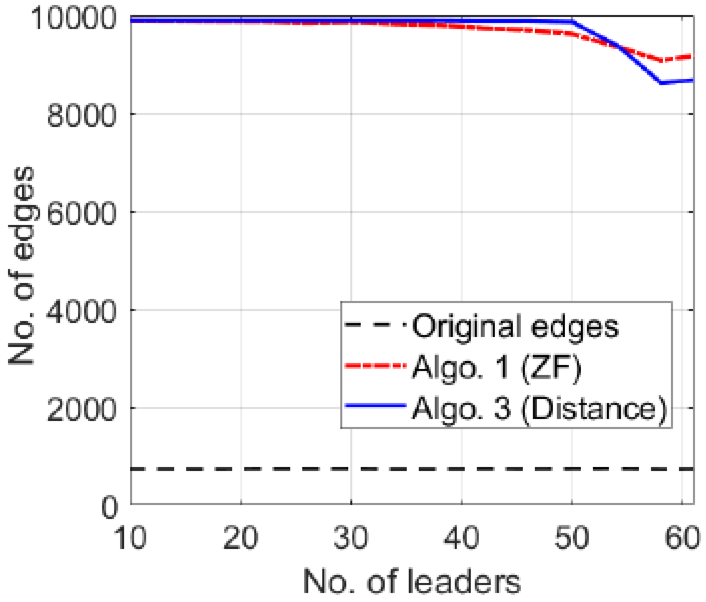
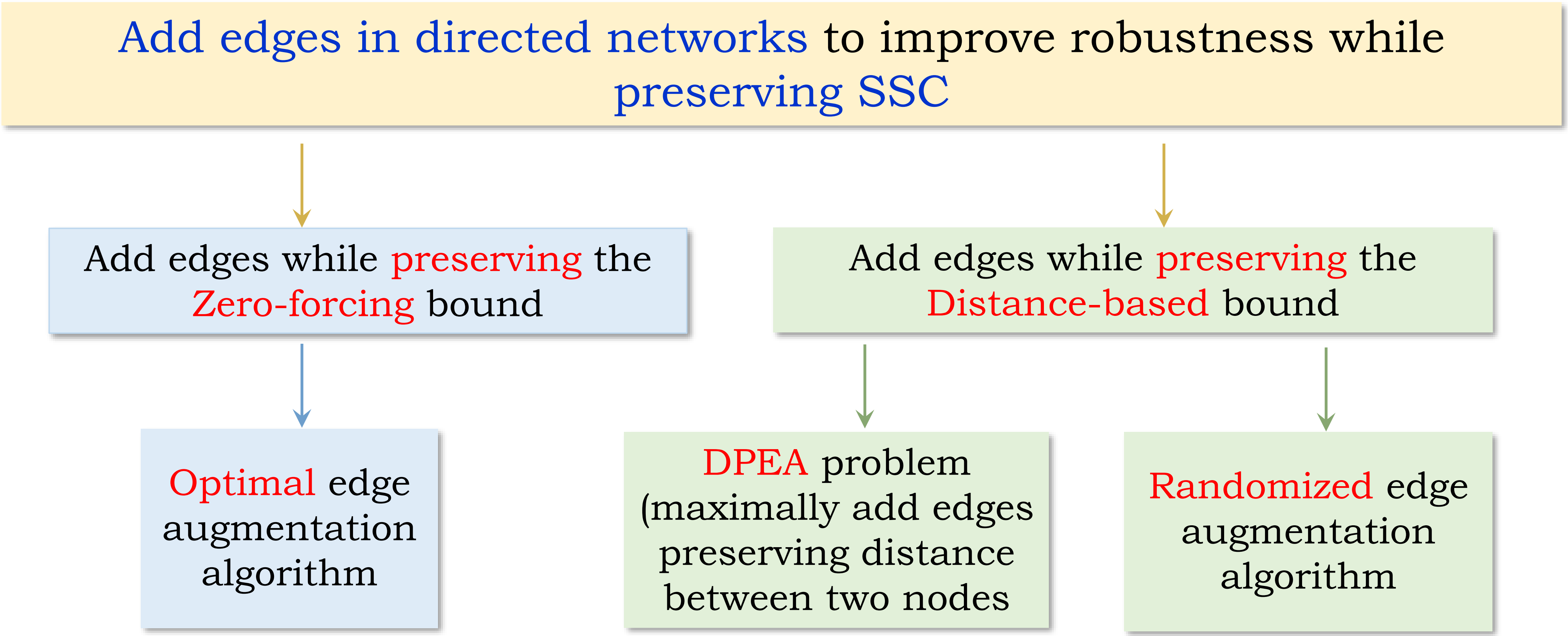


Number of edges added by ZF-based and distance-based augmentation algorithms while preserving ***the same (ZF-based)*** *bound.*

Summarizing ….



Add edges in directed networks to improve robustness while preserving SSC

Add edges while preserving the Zero-forcing bound

Add edges while preserving the Distance-based bound

Optimal edge augmentation algorithm

DPEA problem (maximally add edges preserving distance between two nodes

Randomized edge augmentation algorithm

Thank You

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