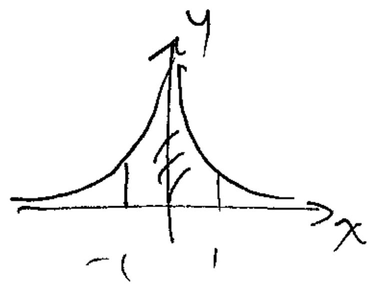
Math 1497 — Calc 2

Consider

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{1} x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{1} = \left.-\frac{1}{x}\right|_{-1}^{1}$$

$$= -\frac{1}{1} + \left(\frac{1}{-1}\right) = -2$$

However $\frac{1}{x^2} \geq 0$



so $\int_a^b \frac{1}{x^2}\,dx \geq 0$ but it's not!

So what happened?

Let's only do ½

$$\int_0^1 \frac{dx}{x^2} = \left.-\frac{1}{x}\right|_0^1 = -1 + \frac{1}{0}$$

? there's the problem.

We integrated over a discontinuity

Consider

$$\int_a^b f(x)\,dx \quad \text{where } f \text{ is discontinuous.}$$

(1) discontinuity at $x = a$

(2) discontinuity at $x = b$

(3) discontinuity inside $[a, b]$ say $x = c$

How do we deal with these – limits

1) $\lim\limits_{m \to a^+} \int_m^b f(x)\,dx$ & see if ~~integral~~ limit exists

if so $\int$ converges

2) $\lim\limits_{M \to b^-} \int_a^M f(x)\,dx$ see if limit exists

(3) split up $\int$ $\quad \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\qquad\qquad\qquad\qquad \#2 \qquad\qquad \#1$

If either $\int$ & limit

does not converge we say the $\int$ diverges

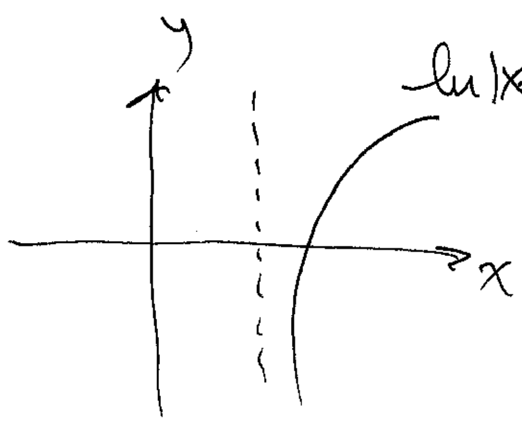
Ex 1 $$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}\int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}\int_a^1 x^{-1/2}\,dx$$

$$= \lim_{a\to 0} 2x^{1/2}\Big|_a^1 = \lim_{a\to 0}\left(2\sqrt{1}-2\sqrt{a}\right) = 2$$

so here the integral converges to 2

Ex 2 $$\int_1^2 \frac{dx}{x-2} = \lim_{b\to 2}\int_1^b \frac{dx}{x-2}$$

$$= \lim_{b\to 2} \ln|x-2|\Big|_1^b = \lim_{b\to 2} \ln|b-2| - \cancelto{0}{\ln(1)}$$



we see from the graph

$$\lim_{b\to 2} \ln|b-2| \to -\infty$$

$x=2$

so here the integral diverges

Ex 3 $\displaystyle\int_{-2}^{3} \frac{dx}{x^3} = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{3} \frac{dx}{x^3}$

we only need to consider 1 if it diverges if it converges we need to check the other integral

$$\int_0^3 \frac{dx}{x^3} = \lim_{a\to 0} \int_a^3 \frac{dx}{x^3} = \lim_{a\to 0} \left.\frac{-1}{2x^2}\right|_a^3$$

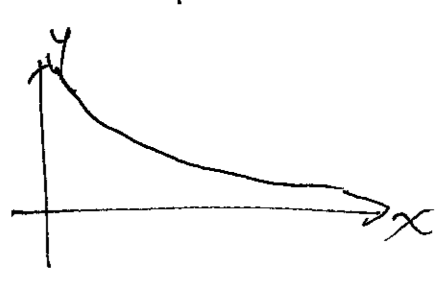
$$= \lim_{a\to 0} -\frac{1}{2\cdot 3^2} + \frac{1}{2a^2} \to \infty$$

so the original integral diverges

This is an example of improper integral

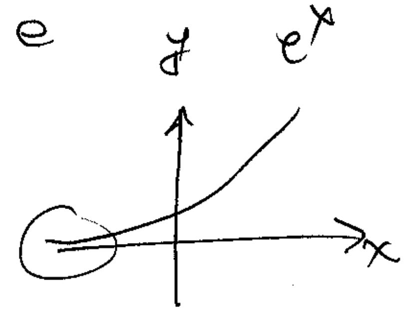
Next Consider

$$\int_1^2 \frac{dx}{x^2} = -\left.\frac{1}{x}\right|_1^2 = -\frac{1}{2}+1 = 1/2$$



ex $$\int_{-\infty}^{0} e^x\,dx = \lim_{a\to-\infty} \int_a^0 e^x\,dx$$

$$= \lim_{a\to-\infty} e^x \Big|_a^0 = \lim_{a\to-\infty} e^0 - e^a$$



$$= 1 - \lim_{a\to-\infty} e^a = 1 - 0 = 1$$

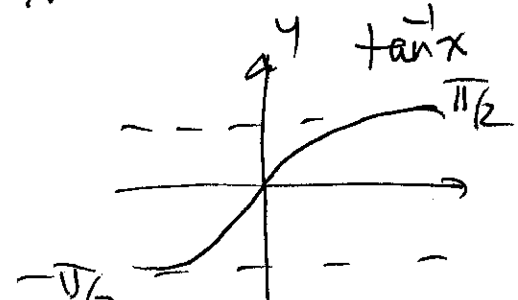
ex $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1}x \Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x \Big|_0^b$$

$$= \lim_{a\to-\infty} \overset{0}{\cancel{\tan^{-1}0}} - \tan^{-1}a + \lim_{b\to\infty} \tan^{-1}b - \overset{0}{\cancel{\tan^{-1}0}}$$

$$= -(-\pi/2) + \pi/2 = \underline{\underline{\pi}}$$



$$\int_1^3 \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^3 = -\frac{1}{3} + 1 = 2/3$$

$$\int_1^4 \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^4 = -\frac{1}{4} + 1 = 3/4$$

$\vdots$

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{b} + 1 \to 1$$

so the integral converges

In general

① $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

② $$\int_{-\infty}^b f(x) = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

③ $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Ex $\int_0^{\infty} \frac{dx}{x^2} = \lim_{\substack{a \to 0 \\ b \to \infty}} \int_a^b \frac{dx}{x^2}$

$= \lim_{\substack{a \to 0 \\ b \to \infty}} -\frac{1}{x}\Big|_a^b = \lim_{\substack{a \to 0 \\ b \to \infty}} -\frac{1}{b} + \frac{1}{a}$

$= \lim_{b \to \infty} -\frac{1}{b} + \lim_{a \to 0} \frac{1}{a}$ ← undefined

so $\int_0^{\infty} \frac{dx}{x^2}$ diverges