Spline Collocation Method for Solving Three Dimensional Parabolic Equations

Abstract
The present work describes spline collocation approximation for solving Heat Conduction In Rectangular problem. The mathematical problem gives rise to solve a linear partial differential equation which is of three dimensional parabolic type. The method involves the solution of algebraic linear equation which can be written in the matrix form is main advantage. The solution are obtained by this method compared with analytic solution to demonstrate the justification and simplicity of the spline approximation converge to analytic solution.

Keywords: Spline Collocation Method, Partial Differential Equation.

1. INTRODUCTION:
Two common questions are encountered while the numerical solution to the problem is obtained. The first is about its acceptance whether it is sufficiently close to the true solution or not. If one has an analytic solution then this can be answered very clearly but in either case it is not so easy. One has to be careful while concluding that a particular numerical solution is acceptable when an analytic solution is not available. Normally a method is selected which requires a minimum number of steps, consuming the shortest computational time and yet one that does not produce an excessive errors.

2. SPLINE COLLOCATION METHOD:
For solving linear and nonlinear differential equation with the help of the numerical methods required much computational work and time. Bickley [1968] Suggested the method of spline function containing truncated power polynomials to solve a linear boundary value problem Ahlberg et al [1967] used cardinal splines for solving differential equations. Doctor et al [1983,1984] have shown that the method of spline collocation is quite useful for the solution of physical phenomena which give rise to linear parabolic one dimensional partial differential equation. The method demonstrates the use of spline function. Spline functions are piecewise polynomial and their successive derivatives are continuous. They were used for data interpolation initially. In 1967, Blue [1969] suggested the use of spline function for the solution of B.V.P.

\[ y'' = f(x,y,y') \] 

with boundary conditions
\[ G_1 [Y(0),Y'(0)] \]
\[ G_2 [Y(1),y'(1)] \] 

(2.1)
(2.2)

The following recurrence relations were used.

\[ S''(x_{i-1}) + 4S''(x_i) + S''(x_{i+1}) = 6/h^2 (f(x_{i-1}) - 2f(x_i) + f(x_{i+1})) \] 

(2.3)
2.1 Spline Formula To Solve Parabolic Partial Differential Equation With Three Space Variables

The general form of linear parabolic partial differential equation with three space variables is of the form

$$u_t = R^2 (u_{xx} + u_{yy} + u_{zz}) ; 0 < x < a, 0 < y < b, 0 < z < c$$  \hspace{1cm} (2.1.1)

With Dirichlet condition prescribed on the boundaries $x=0,y=0,z=0$. We should divide the region $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$ in to $L,M$ and $N$ sub-intervals of width $\Delta x$, $\Delta y$, $\Delta z$ respectively such that $L\Delta x = a$, $M\Delta y = b$, and $N\Delta z = c$. Let us denote the points of subdivisions by $x_i : i = 0,1,2,\ldots,L$, $y_j : j = 0,1,2,\ldots,M$, $z_k : k = 0,1,2,\ldots,N$. Let us $u_{i,j,k,p}$ denotes the values of $u$ at the $(i,j,k)^{th}$ mesh point at the time $p \Delta t$. For simplicity, let us take cubic region with equal sides and $\Delta x = \Delta y = \Delta z = h$ (say) and $L=M=N$.

Approximate the function $u$ at time $p \Delta t$ by cubic spline $S(x)$. Discretizing the left side of equation (2.1.1) by forward difference formula and replacing right hand side by three times of the second derivative. i.e. $3S''(x_{i,k})$ at $p^{th}$ level we get

$$(u_{i,j,k,p+1} + 4u_{i,j,k,p} + u_{i+1,j,k,p+1}) = (1+18r)u_{i-1,j,k,p} + (4-36r)u_{i,j,k,p} + (1+18r)u_{i+1,j,k,p}$$

i,j,k=1,2,\ldots,N-1  \hspace{1cm} (2.1.2)

Where $r = R^2 \Delta t / h^2$. These set of $(N-1) \times (N-1) \times (N-1)$ equations in $(N-1) \times (N-1) \times (N-1)$ unknowns can be solved by any well-known method. The above set of simultaneous equations gives a square matrix. The above equation (2.1.2) is known as cubic spline explicit formula to solve equation (2.1.1). For this explicit method maximum possible value of $r$, for stability and convergence of this method, maximum possible value of $r$ is $r \leq 1/6$.

In Implicit method, he finite difference replacement of equation (2.1.1) is

$$\frac{u_{i,j,k,p+1} - u_{i,j,k,p}}{\Delta t} = R^2 \left[ \frac{3}{2} (S''_{i,j,k,p} + S''_{i,j,k,p+1}) \right]$$

Where $S''_{i,j,k,p}$ and $S''_{i,j,k,p+1}$ denote the second derivatives of $S(x)$ at $x = x''_{i,j,k,p}$ at the time interval $p$ and $p+1$ respectively, we get

$$(1-9r)u_{i,j,k,p+1} + (4+18r)u_{i,j,k,p} + (1-9r)u_{i+1,j,k,p+1} = (1+9r)u_{i-1,j,k,p} + (4-18r)u_{i,j,k,p} + (1+9r)u_{i+1,j,k,p}$$

Where $r = R^2 \Delta t / h^2$ and $i,j,k=1,2,3,\ldots,N-1$

Above relation known as cubic spline implicit formula to solve equation (2.1.1). Like explicit scheme we get $(N-1) \times (N-1) \times (N-1)$ equation in $(N-1) \times (N-1) \times (N-1)$ unknowns. These simultaneous equations with square matrix can be solved any standard method. In both methods once the value of $u$ are known at $(p+1)^{th}$ level, we can proceed to compute the next level $(p+2)$ by same techniques described as above.

3. HEAT CONDUCTION IN THIN RECTANGULAR VOLUME:

Consider a flow of heat in a rectangular volume with sides of length $\Delta x, \Delta y, \Delta z$ along the co-ordinate axes $x,y,z$ (figure(3.1))
The amount of heat entering the element through the face PQRS in time $\Delta t$ is
$$-k(\Delta y \Delta z)(\partial u/\partial x)_x \Delta t$$
in the element through the approximate face is
$$-k(\Delta y \Delta z)(\partial u/\partial x)_x \Delta t$$
Where $k$ is the thermal conductivity of the material and $u(x,y,z,t)$ is the temperature function. The negative signs are taken because the heat flows in the direction of decreasing temperature. Hence the quantity of heat remaining in the solid as a result of entry through the PQRS and exit through the opposite face is
$$-\{\left(\frac{\partial^2 u}{\partial x^2}\right)_x \Delta x\} \Delta y \Delta z \Delta t$$
(3.1)
Up to a first approximation.
Similarly, the corresponding differences in the heat entering and leaving through the nearing two pairs of opposite face are
$$k\{\left(\frac{\partial^2 u}{\partial y^2}\right)_y \Delta y \Delta z \Delta x$$
and $k\{\left(\frac{\partial^2 u}{\partial z^2}\right)_z \Delta z \Delta x \Delta y \Delta t$$
(3.2)
Hence the total heat retained by the solid in time $\Delta t$ is the sum of (3.1) and (3.2) which is equal to the heat required to raise the temperature to raise the temperature of the element by $\Delta u$. Thus we have
$$k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2)\Delta x \Delta y \Delta z$$
$$= (\rho \Delta x \Delta y \Delta z) s \Delta u$$
(3.3)
Where $\rho$ is density and $s$ be the specific heat of the solid. Dividing by $\Delta x \Delta y \Delta z$ and limit as $\Delta t \to 0$, we get
$$\partial u/\partial t = R^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial x^2 + \partial^2 u/\partial x^2); R^2 = k/s \rho$$
(3.3)
Above equation (3.3) is the heat equation in three space variables. Also it is known as parabolic partial differential equation with three space variables $x,y,z$ and time variable $t$. Solution of above equation (3.3) gives the temperature distribution in the rectangular volume.
Consider the edges of rectangular volume of side 1 are kept at temperature zero and faces are perfectly insulated. Hence the flow of heat in the volume governed by the equation (3.3) with boundary conditions
$$u(x,0,z,t)=0, u(x,y,0,t)=0, u(0,y,z,t)=0$$
(3.4)
Where \( \theta \leq x, y, z \leq l \)

Let initial temperature distribution in the volume be
\[
u(x, y, z, 0) = \sin \Pi x \sin \Pi y \sin \Pi z
\]  
(3.5)

Where \( 0 \leq x, y, z \leq 1 \)

The given Problem with boundary and initial conditions described as above is solved by explicit and implicit method.

### 4.1 SPLINE SOLUTION WITH EXPLICIT METHOD

We shall determine the solution of equation (3.3) satisfying the initial and boundary conditions defined by the equation (3.4) and (3.5) respectively, using the explicit formula given by equation(3.3) as follows

For that let \( h=1/10 \) \( R^2 = 0.001 \) and \( \Delta t = 0.001 \) which gives \( r=0.001 \) \( 1+18r=1.018 \) and \( 4-36r = 3.964 \)

Substituting the value of \( 1+18r \) \( 4-36r \) with initial and boundary condition in equation (3.3) gives following result

For \( p=0, k=1 \) and \( j=1 \)

\[
i=1 \quad u_{0,1,1,1} + 4u_{1,1,1,1} + u_{2,1,1,1} = 0.17411
\]

Since \( u_{0,1,1,1} = 0 \)

\[
4u_{1,1,1,1} + u_{2,1,1,1} = 0.17411
\]

\[
i=2 \quad u_{1,1,1,1} + 4u_{2,1,1,1} + u_{3,1,1,1} = 0.331177
\]

\[
i=3 \quad u_{2,1,1,1} + 4u_{3,1,1,1} + u_{4,1,1,1} = 0.455827
\]

etc.

Proceeding in this way for \( i,j,k = 1,2,3,...,9 \)

We get 9x9x9 equations in 9x9x9 unknowns, which can be solved by any standard method once the values of \( p=0 \) are obtained, the results for \( p=1,2.. \) are obtained by same process discussed as above . Due to symmetry of the solution the result obtained for \( 0 \leq z \leq 0.5, 0 \leq y \leq 0.5, 0 \leq x \leq 0.5 \) at \( t=0.001 \) and \( t=0.003 \) Hear the result for \( z=0.5, y=0.1 \) at \( t=0.001 \) are given in table 1 and plotted in fig. 1

### 4.2 SPLINE SOLUTION WITH IMPLICIT METHOD:

Using implicit formula given by equation(2.1.4)the solution of equation (3.3) satisfying boundary and initial conditions For \( p=0, k=1,j=1 \)

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

Similarly in this way we get 9x9x9 simultaneous equations we get the temperature distribution in rectangular volume at time \( t=0.001 \) Proceeding in the same way we get the temperature distribution at \( t=0.001,0.003 \) etc. Result for \( z=0.5, y=0.1 \) at \( t=0.003 \) are given in table 6.1 and plotted in fig.6.1

### 5. RESULT

#### TABLE 5.1

<table>
<thead>
<tr>
<th>Temperature Distribution In Rectangular Volume U</th>
</tr>
</thead>
<tbody>
<tr>
<td>At ( t=0.001 ), ( z=0.5 )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>( X, Y )</td>
</tr>
<tr>
<td>0.0</td>
</tr>
</tbody>
</table>
TABLE 5.2
Temperature Distribution In Rectangular Volume U

<table>
<thead>
<tr>
<th>X, Y</th>
<th>0.0</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.10</td>
<td>0.095406</td>
<td>0.181473</td>
<td>0.345182</td>
<td>0.475102</td>
<td>0.558516</td>
<td>0.587259</td>
</tr>
<tr>
<td>0.20</td>
<td>0.181473</td>
<td>0.345182</td>
<td>0.475102</td>
<td>0.653922</td>
<td>0.768732</td>
<td>0.808292</td>
</tr>
<tr>
<td>0.30</td>
<td>0.249776</td>
<td>0.475102</td>
<td>0.653922</td>
<td>0.768732</td>
<td>0.903699</td>
<td>0.950205</td>
</tr>
<tr>
<td>0.40</td>
<td>0.293630</td>
<td>0.558516</td>
<td>0.768732</td>
<td>0.903699</td>
<td>0.950205</td>
<td>0.999105</td>
</tr>
<tr>
<td>0.50</td>
<td>0.308740</td>
<td>0.587259</td>
<td>0.808292</td>
<td>0.950205</td>
<td>0.999105</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 5.3
Error Analysis Of Temperature Distribution Rectangular Volume U at t=0.003 , z=0.5 y=0.1

| X  | UEXT | |UEXT−UE|X10⁻⁶ | |UEXT−UI|X10⁻⁶ |
|----|------|----------------|-------------|----------------|
| 0.0 | 0.000000 | 0 | 0 |
| 0.10 | 0.095407 | 1 | 1 |
| 0.20 | 0.181474 | 1 | 1 |
| 0.30 | 0.249778 | 2 | 2 |
| 0.40 | 0.293632 | 2 | 2 |
| 0.50 | 0.308743 | 3 | 3 |

Fig (5.1)
Temperature Distribution In Rectangular Volume through cubic spline explicit method (at z=0.5,y=0.1)
6. CONCLUSION:

The solution of equation (3.3) obtained by spline explicit and implicit as well as explicit methods are compared with the exact solutions in table 5.3. Fig. 5.3 indicates the error analysis which compares the exact solutions with spline solutions obtained by both explicit as well as implicit method. From the fig. 5.3 and table 5.3 it is clear that spline solutions are quite reliable up to five digits of decimal places.

7. REFERENCES:


AUTHOR’S BIOGRAPHY:

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