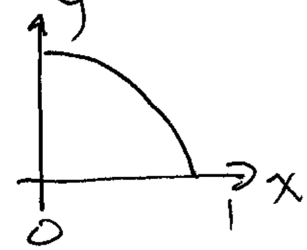
Math 1497 Calc 2

Earlier we saw that the area under $y = \sqrt{1-x^2}$ is given by



$$A = \int_0^1 \sqrt{1-x^2}\,dx$$

so how do we integrate this?

Today, we will look at another type of substitution called a trig sub and we be used for integrals of the form

$$\int \sqrt{a^2-x^2}\,dx \qquad \int \sqrt{a^2+x^2}\,dx \qquad \int \sqrt{x^2-a^2}\,dx$$

and variation thereof. The key idea here is to use the identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1+\tan^2\theta = \sec^2\theta$$

or $1-\sin^2\theta = \cos^2\theta, \quad 1+\tan^2\theta = \sec^2\theta \quad \sec^2\theta - 1 = \tan^2\theta$

we will consider by example

ex 1 $\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{-\frac{1}{2}\,du}{u^{1/2}} = -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2}+C$

we could use the u sub $u = 1-x^2$ $du = -2x\,dx$

so $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}+c$

instead we will use the trig sub

$x = \sin\theta$ $1-x^2 = 1-\sin^2\theta = \cos^2\theta$

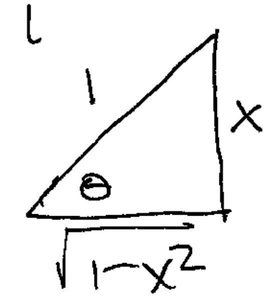
so $dx = \cos\theta\,d\theta$ $\sqrt{1-x^2} = \cos\theta$

so $\int \frac{\sin\theta \cdot \cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$ a trig integral

$= \int \sin\theta\,d\theta = -\cos\theta + c$

Now to get back in terms of x

$\frac{x}{1} = \sin\theta$



$\cos\theta = \frac{a}{h} = \frac{\sqrt{1-x}}{1} = -\sqrt{1-x^2}+c$

seen earlier

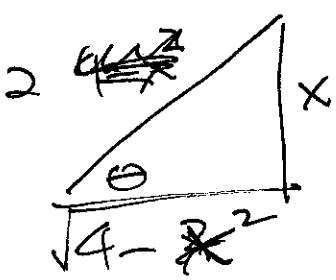
ex 2 $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$ Use $x = 2\sin\theta$

Since $4-x^2 = 4-4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta$

also $dx = 2\cos\theta\,d\theta$ so our $\int$ becomes

$$\int \frac{2\cos\theta\cdot 2\cos\theta\,d\theta}{2\sin\theta} = 2\int \frac{\cos^2\theta\,d\theta}{\sin\theta} = 2\int \frac{1-\sin^2\theta\,d\theta}{\sin\theta}$$

$$2\int \csc\theta - \sin\theta\,d\theta = 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C$$



$\sin\theta = \frac{x}{2}$ so $\csc\theta = \frac{1}{\sin\theta} = \frac{2}{x}$

$\cot\theta = \frac{\sqrt{4-x^2}}{x}$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \cancel{2}\cdot\frac{\sqrt{4-x^2}}{\cancel{2}} + C$$

<u>ex 3</u> $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}}$

Here we use $x = 3\tan\theta$

$\therefore \quad 9+x^2 = 9+9\tan^2\theta = 9(1+\tan^2\theta) = 9\sec^2\theta$

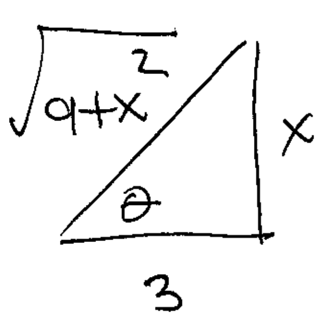
so $(9+x^2)^{3/2} = (9\sec^2\theta)^{3/2} = (3^2\sec^2\theta)^{3/2} = 3^3\sec^3\theta$

$dx = 3\sec^2\theta\, d\theta$

& our $\int$ becomes

$$\int \frac{3\sec^2\theta}{3^3\sec^3\theta}\, d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta + c = \frac{1}{9}\cdot\frac{x}{\sqrt{9+x^2}} + c$$



Ex 4 $\int \frac{\sqrt{x^2-1}}{x}\,dx$

Here we use $x = \sec\theta$ $\therefore$ $x^2-1 = \sec^2\theta - 1 = \tan^2\theta$

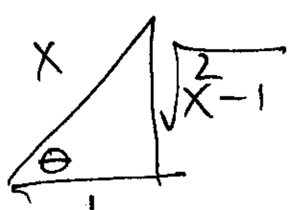
$dx = \sec\theta\tan\theta\,d\theta$

$$\int \frac{\tan\theta \cdot \cancel{\sec}\theta\,\tan\theta\,d\theta}{\cancel{\sec}\theta}$$

$\int \tan^2\theta\,d\theta = \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + c$

$= \sqrt{x^2-1} - \sec^{-1}(x) + c$



$\sec\theta = \frac{x}{1}$

$\cos\theta = \frac{1}{x}$

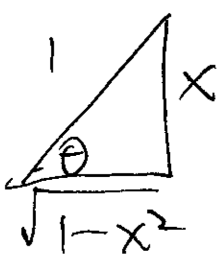
ex 5 $\int \sqrt{1-x^2}\,dx$

$x=\sin\theta$ $\quad dx=\cos\theta\,d\theta$ $\quad \sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$

$\int \cos^2\theta\,d\theta$ use double angle formula

$$\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2}+\frac{\sin 2\theta}{4}+c$$

$$= \frac{\theta}{2}+\frac{2\sin\theta\cos\theta}{4}+c$$



$\sin\theta=\frac{x}{1}$ $\quad \cos\theta=\frac{\sqrt{1-x^2}}{1}$

$$= \frac{1}{2}\sin^{-1}x+\frac{x\sqrt{1-x^2}}{2}+C$$