Math 4315 - PDE's

Consider the PDE

$$u_x - u_y = 1, \quad u(x,x) = 2x. \qquad (1)$$

First consider the change of variables

$$r = \frac{x+y}{2}, \quad s = \frac{x-y}{2}. \qquad (2)$$

From the chain rule

$$u_x = u_r r_x + u_s s_x, \qquad u_y = u_r r_y + u_s s_y$$
$$= \frac{1}{2}u_r + \frac{1}{2}u_s \qquad\qquad = \frac{1}{2}u_r - \frac{1}{2}u_s$$

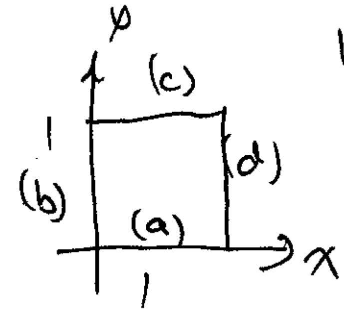
so the PDE becomes

$$u_x - u_y = 1$$

$$\frac{1}{2}u_r + \frac{1}{2}u_s - \frac{1}{2}u_r + \frac{1}{2}u_s = 1$$

$$\Rightarrow u_s = 1 \quad (\text{easier}) \qquad (3)$$

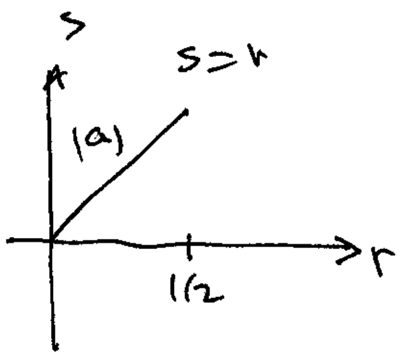
we now determine what happens to the (x,y) plane under the change of variables (CofV) given in (2). Here we consider the unit square.

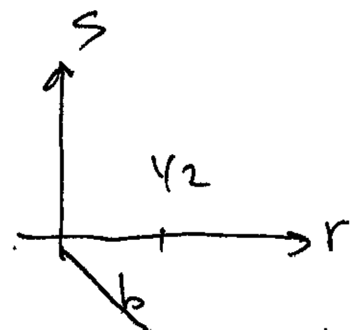


we label the side (a) (b) (c) (d) and look at each separately

(a) Here $y=0$ $x=0\to 1$ from (2) $r=\frac{x}{2}$, $s=\frac{x}{2}$

so we take $s=r$ & $r: 0\to \frac{1}{2}$





(b) Here $x=0$ so from (2) $r=\frac{y}{2}$, $s=-\frac{y}{2}$

so $s=-r$ & $r: 0\to 1$ $\because y: 0\to 1$

(c) $y=1$, $x: 0\to 1$

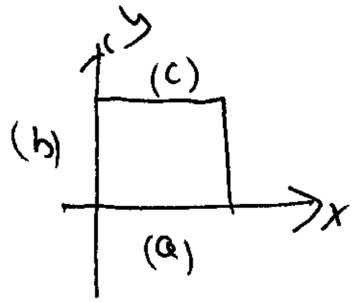
so $r=\frac{x+1}{2}$, $s=\frac{x-1}{2}$

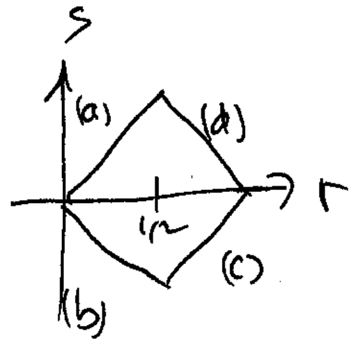
$\Rightarrow$ $s=r-1$

(d) $x=1$, $y: 0\to 1$

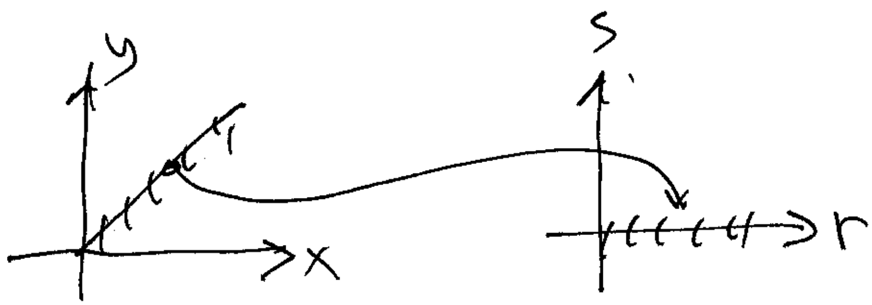
$r=\frac{1}{2}+\frac{y}{2}$ $s=\frac{1}{2}-\frac{y}{2}$

$s=-r+1$





So we see the region changes. What's important to note that the boundary $y = x$ in (1) becomes $s = 0$ & $r = x$



If $x = r$ then $y = x = r$ & $u = 2x = 2r$

so in the (r, s) plane we have

$$u_s = 1 \qquad u\big|_{s=0} = 2r$$

and we can solve this

$$u = s + c(r)$$

$$s = 0 \;\; u = 2r \Rightarrow u = s + 2r$$

From (2) we substitue so

$$u = \frac{x-y}{2} + 2\left(\frac{x+y}{2}\right)$$

$$= \frac{3x}{2} + y/2 \quad \Leftarrow \text{the sol}^n$$

let us return to (1)

$$u_x - u_y = 1 \qquad u(x,x) = 2x$$

From the chain rule

$$u_s = u_x x_s + u_y y_s$$

we choose x_s & y_s to have $u_x - u_y = 1$

so

$x_s = 1$ $\quad$ $s=0$, $x=r$ $\quad$ we choose the new boundary in the (r,s) plane

$y_s = -1$ $\quad$ $y=r$ $\quad$ as $s=0$ & pick $x=r$

$u_s = 1$ $\quad$ $u=2r$ $\quad$ so $y=x=r$, $u=2r$

we now solve these

$x_s = 1 \Rightarrow x = s + a(r) \quad s = 0 \; x = r \Rightarrow a(r) = r$

so $\boxed{x = s + r}$

$y_s = -1 \Rightarrow y = -s + b(r) \quad s = 0 \; y = r \Rightarrow b(r) = r$

so $\boxed{y = -s + r}$

$u_s = 1 \Rightarrow u = s + c(r) \quad s = 0 \; u = 2r \Rightarrow c(r) = 2r$

so $\boxed{u = s + 2r}$ ← we just saw this

Now $\left.\begin{array}{l} x = s + r \\ y = -s + r \end{array}\right\} \Rightarrow r = \frac{x+y}{2}, \quad s = \frac{x-y}{2}$

which is exact the Cof v in (2)

so by choosing

$x_s = 1, \; y_s = -1$ with $s = 0 \; x = r, \; y = r$

gives the change of variable automatically