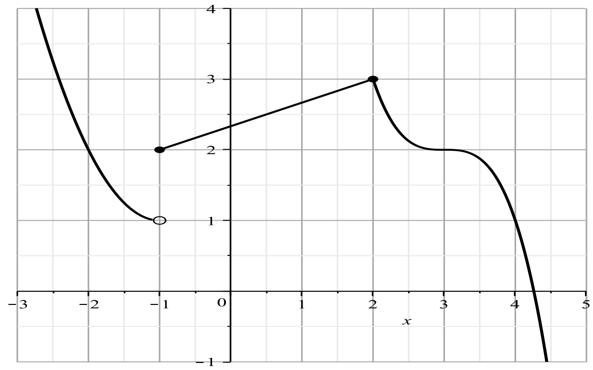
Math 1496 - Sample Test 1 Solutions

1. From the following graph determine the following limits.

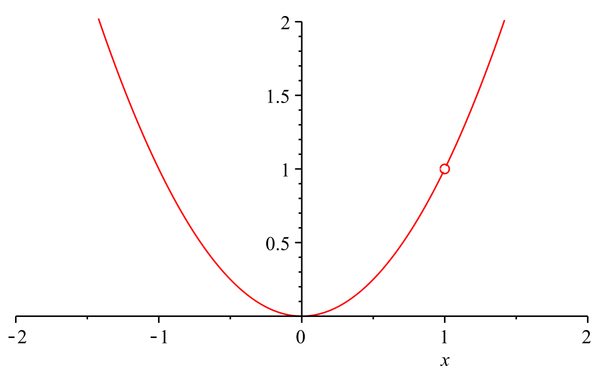


$$(i)\quad \lim_{x\to -1^-} f(x) = 1 \qquad (ii)\quad \lim_{x\to -1^+} f(x) = 2 \qquad (iii)\quad \lim_{x\to -1} f(x) = \text{DNE}$$

$$(iv)\quad \lim_{x\to 2^-} f(x) = 3 \qquad (v)\quad \lim_{x\to 2^+} f(x) = 3 \qquad (vi)\quad \lim_{x\to 2} f(x) = 3$$

2. Calculate $\lim_{x\to 1} \dfrac{x^3 - x^2}{x - 1}$ using the techniques of graphically, numerically and analytically.

(i) Graphically



from which the graph says $\lim_{x\to 1} \dfrac{x^3 - x^2}{x - 1} = 1.$

(ii) Numerically

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.8100	0.9801	0.9980	1.0020	1.0201	1.2100

from which the table says $\lim_{x\to 1} \dfrac{x^3 - x^2}{x - 1} = 1.$

(iii) Analytically

$$\lim_{x\to 1}\frac{x^3-x^2}{x-1}=\lim_{x\to 1}\frac{x^2\cancel{(x-1)}}{\cancel{x-1}}=\lim_{x\to 1}x^2=1$$

2. Calculate the following limits analytically.

$$(i)\quad \lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to 4}\frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}}=\lim_{x\to 4}\sqrt{x}+2=4$$

$$(ii)\quad \lim_{x\to 0}\frac{\sin 4x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin 4x}{x}}{\frac{\sin 2x}{x}}=\frac{4}{2}=2\quad\text{since}\quad \lim_{x\to 0}\frac{\sin mx}{x}=m$$

$$(iii)\quad \lim_{x\to\infty}\frac{3x^2+4}{x^2+2x+1}=\lim_{x\to\infty}\frac{3+\frac{4}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}}=\frac{3+0}{1+0+0}=3$$

3. Calculate the first derivative (either $f'(x)$ or y') of the following. Do not simplify your answer

$$(i)\quad y=\frac{4e^x}{x^2+1},\quad y'=\frac{4e^x(x^2+1)-4e^x\cdot 2x}{(x^2+1)^2}$$

$$(ii)\quad y=x^2\tan x,\quad y'=2x\tan x+x^2\sec^2 x$$

$$
\begin{aligned}
(iii)\quad f(x) &= \sin\left(\sqrt{4x^2+x}\right),\\
f'(x) &= \cos\left(\sqrt{4x^2+x}\right)\cdot\frac{1}{2}\left(4x^2+x\right)^{-1/2}\cdot(8x+1)
\end{aligned}
$$

4. The definition

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{n}\qquad(1)$$

If $f(x)=3x^2-5x+2$ then

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{3(x+h)^2-5(x+h)+2-(3x^2-5x+2)}{h}\\
&= \lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2-\cancel{5x}-5h+\cancel{2}-\cancel{3x^2}+\cancel{5x}-\cancel{2}}{h}\\
&= \lim_{h\to 0}\frac{\cancel{h}(6x+2h-5)}{\cancel{h}}=\lim_{h\to 0}6x+2h-5=6x-5
\end{aligned}
$$

5. If $y = x^4 - 2x^3 + 2x^2$ then $y' = 4x^3 - 6x^2 + 4x$. At $x = 1, y = 1$ and $y'|_{x=1} = 4 - 6 + 4 = 2$ so the equation of the tangent is $y - 1 = 2(x - 1)$.

6. If

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x^3 & x > 0 \end{cases}$$

is $f(x)$ continuous and differentiable at $x = 0$?

Part (i)

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} x^2 = 0$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x^3 = 0$$

Further $f(0) = 0^2 = 0$. Since

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0,$$

$f(x)$ is continuous at $x = 0$.

Part (ii)

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$$
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x^3}{x} = \lim_{x \to 0} x^2 = 0$$

Since

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0,$$

then $f(x)$ is differentiable at $x = 0$ an we define $f'(0) = 0$.

7. Prove

$$\lim_{x \to 2} 2x - 1 = 3$$

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|(2x - 1) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta$$

Aside			Proof		
$\mid (2x-1)-3 \mid$	$<$	ε	$\mid x-2 \mid$	$<$	$\dfrac{\varepsilon}{2}$
$\mid 2x-4 \mid$	$<$	ε	$2 \mid x-2 \mid$	$<$	ε
$2 \mid x-2 \mid$	$<$	ε	$\mid 2x-4 \mid$	$<$	ε
$\mid x-2 \mid$	$<$	$\varepsilon/2$	$\mid (2x-1)-3 \mid$	$<$	ε
pick $\delta = \varepsilon/2$					

Therefore

$$|(2x-1)-3| < \varepsilon \quad \text{whenever} \quad 0 < |x-2| < \varepsilon/2$$