Math 6345 – Adv. ODEs

Complex Eigenvalues

$$\dot{\bar{x}} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \bar{x}$$

$$\det|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & 1 \\ -4 & \lambda \end{vmatrix} = 0 \qquad \lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$\lambda = -2i$

$$\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad -2iu + v = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$$

$\lambda = 2i$

$$\begin{pmatrix} 2i & 1 \\ -4 & 2i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad 2iu + v = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$$

Sol^n

$$\bar{x} = k_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i \right] e^{-2it} + k_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} i \right] e^{2it}$$

$$= c_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t \right] + c_2 \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t \right]$$

where $c_1 = k_1 + k_2 \qquad c_2 = i(k_1 - k_2)$

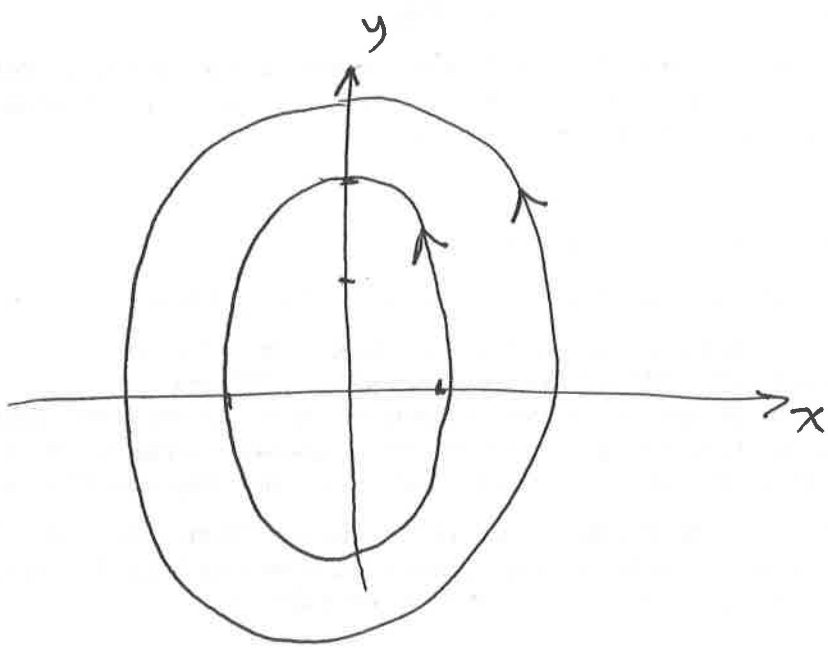
Expanding gives

$$x = c_1 \cos 2t - c_2 \sin 2t$$

$$y = 2c_1 \sin 2t + 2c_2 \cos 2t$$

and eliminating t gives

$$x^2 + \frac{y^2}{4} = c_1^2 + c_2^2 \quad \text{ellipses}$$



to find the direction we go to the ODEs

$$\dot{x} = -y, \quad \dot{y} = 4x$$

if $x, y > 0$ $\dot{x} < 0$, $\dot{y} > 0$

Ex $\frac{d\bar{x}}{dt} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \bar{x}$

$\det|\lambda I - A| = 0 \quad \begin{vmatrix} \lambda+1 & -2 \\ 2 & \lambda+1 \end{vmatrix} = 0 \quad \lambda^2 + 2\lambda + 1 + 4 = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$

$\lambda = -1 - 2i \qquad = -1 \pm 2i$

$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -2iu - 2v = 0$

$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$

$\bar{x}_1 = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i \right] e^{-t} (\cos 2t - i \sin 2t)$

$= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right] e^{-t}$

$- i \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t \right] e^{-t}$

so the 2 sol^{n}'s are the real & imagery parts

$\bar{x} = c_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right] e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t \right] e^{-t}$

So

$$x = c_1 \cos 2t\, e^{-t} + c_2 \sin 2t\, e^{-t}$$

$$y = -c_1 \sin 2t\, e^{-t} + c_2 \cos 2t\, e^{-t}$$

So

$$x^2 + y^2 = (c_1^2 + c_2^2)\, e^{-2t}$$

without the exponential term these would be circles. With the exponential term the trajectories decay to the origin