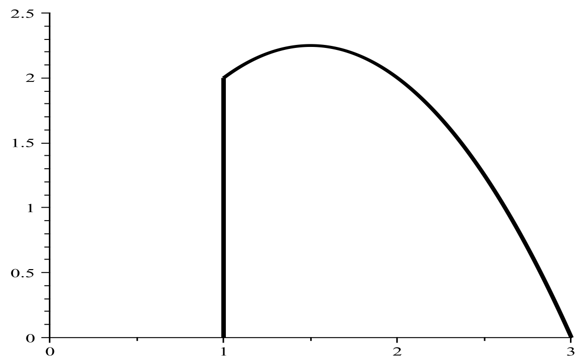
Math 1496 - Sample Test 3 - Solns

1. Using n rectangles and the limit process, find the area under the given curve.

$$y = 3x - x^2 \text{ on } [1,3]$$



Sol: The thickness of each rectangle is $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$. We choose $x_i^* = 1 + \dfrac{2i}{n}$ so the height of the i^{th} rectangles is $h_i = f(x_i^*) = 3\left(1 + \dfrac{2i}{n}\right) - \left(1 + \dfrac{2i}{n}\right)^2$. Next, the area of this rectangle is $A_i = f(x_i^*)\Delta x = \left[3\left(1 + \dfrac{2i}{n}\right) - \left(1 + \dfrac{2i}{n}\right)^2\right]\dfrac{2}{n}$

Thus,

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(1 + \frac{2i}{n}\right) - \left(1 + \frac{2i}{n}\right)^2\right]\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} + \frac{4i}{n^2} - \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\cdot n + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
&= 4 + 2 - \frac{8}{3} = \frac{10}{3}
\end{aligned}$$

2a. A manufacturer wants to design a box with an **open** top having a square base and an area of 27 sq. inches. What dimensions will produce a box with maximum volume?

Sol: First draw and label a picture (not included here). If we denote the side of the square base by x and the height y, the volume of the box is $V = x^2y$ and the area $A = x^2 + 4xy = 27$. Solving the latter for y gives

$$y = \frac{27 - x^2}{4x} \tag{1}$$

so the volume becomes

$$V = x^2 \cdot \frac{27 - x^2}{4x} = \frac{27x - x^3}{4} \tag{2}$$

Now

$$V' = \frac{27 - 3x^2}{4} \tag{3}$$

and $V' = 0$ when $x = \pm 3$ from which we take the positive case. Since $V'' = 3x/2 < 0$ when $x = 3$ we have a maximum. With $x = 3$, then $y = \dfrac{27 - 3^2}{4 \cdot 3} = \dfrac{3}{2}$. The dimensions are $3'' \times 3'' \times 3/2''$.

2b. A rectangular dog pen is being built against the side of a house using 100 ft of fencing for the remaining 3 sides. What is the maximum area?

Sol: First draw and label a picture (not included here). If we denote the sides of the pen by x and y then the area is $A = xy$ and the fence is $P = 2x + y = 100$. Solving the latter for y gives

$$y = 100 - 2x \tag{4}$$

so the volume becomes

$$A = x(100 - 2x) = 100x - 2x^2 \tag{5}$$

Now

$$A' = 100 - 4x \tag{6}$$

and $A' = 0$ when $x = 25$. Since $A'' = -4 < 0$ we have a maximum. With $x = 25$, then $y = 100 - 2(25) = 50$. The dimensions are $25' \times 50'$.

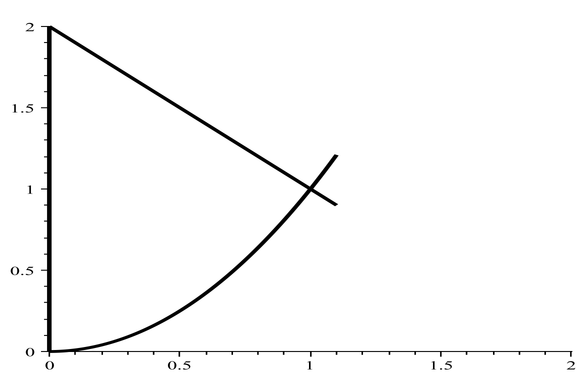
3. Find the area bound by the following curves

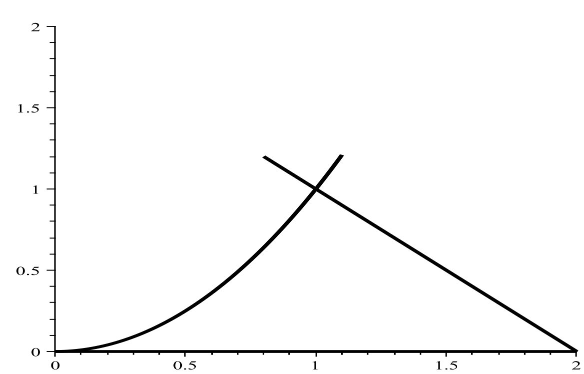
$$y = x^2 \quad y = 2 - x, \quad x = 0, \quad x, y \geq 0.$$

We sketch the curves to find the region of interest. The intersection points between the two curves are

$$x^2 = 2 - x \;\Rightarrow\; x^2 + x - 2 = 0 \;\Rightarrow\; (x+2)(x-1) = 0 \;\Rightarrow\; x = 1, -2$$

and only $x = 1$ is applicable.





The area is then given by

$$A = \int_0^1 \left(2 - x - x^2\right) dx = 2x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$$

Often one mistakes the region and calculates the other region so we'll do it here. Using vertical rectangles, we'll need two integrals so

$$
\begin{aligned}
A &= \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx \\
&= \left.\frac{x^3}{3}\right|_0^1 + \left.\left(2x - \frac{x^2}{2}\right)\right|_1^2 \\
&= 1/3 + ((4-2) - (2-1/2)) = 5/6
\end{aligned}
$$

Using horizontal rectangle we note the intersection point of $x = 1$ which gives $y = 1$ and so the area is

$$
A = \int_0^1 (2 - y - \sqrt{y})\ dy = 2y - \frac{y^2}{2} - \left.\frac{2}{3}y^{3/2}\right|_0^1 = 5/6
$$

4. For the given $y = f(x)$ function and point $x = a$ calculate both dy and Δy.

$$
\begin{aligned}
&(i)\ f(x) = x^2, && x = 2, \quad dx = \Delta x = .1 \\
&(ii)\ f(x) = x^3 - x + 1, && x = 1, \quad dx = \Delta x = .05
\end{aligned}
$$

Sol:
$4(i)$ $\Delta y = f(2.1) - f(2) = (2.1)^2 - 2^2 = .41$ $dy = f'(x)dx = 2xdx$ and when $x = 2$ and $dx = .1$ gives $dy = .4$.

$4(ii)$ $\Delta y = f(1.05) - f(1) = \left(1.05^3 - 1.05 + 1\right) - \left(1^3 - 1 + 1\right) = .1076$ $dy = f'(x)dx = \left(3x^2 - 1\right) dx$ and when $x = 1$ and $dx = .05$ gives $dy = .10$.

5. Evaluate the following

$$
(i)\ \frac{d}{dx}\int_1^x \sin\left(t^2\right)\ dt = \sin\left(x^2\right)
$$

$$
\begin{aligned}
(ii)\ \frac{d}{dx}\int_x^{x^2} \sqrt{1+t^2}\,dt &= \frac{d}{dx}\int_x^0 \sqrt{1+t^2}\,dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}\,dt \\
&= -\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}\,dt \\
&= -\sqrt{1+x^2} + \sqrt{1+(x^2)^2}\cdot 2x
\end{aligned}
$$

6. Evaluate the following indefinite integrals

(i) $\displaystyle\int \sec^2 x \tan x\,dx$

Let $u = \sec x$ so $du = \sec x \tan x\,dx$ and the integral becomes

$$
\int u\,du = \frac{u^2}{2} + c = \frac{\sec^2 x}{2} + c
$$

(ii) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

Let $u = \dfrac{1}{x}$ so $du = -\dfrac{1}{x^2}\,dx$ and the integral becomes

$$\int -e^u\,du = -e^u + c = -e^{1/x} + c$$

(iii) $\displaystyle\int \frac{x}{(x+1)^2}\,dx$

Let $u = x + 1$ so $du = dx$ and the integral becomes

$$\int \frac{u-1}{u^2}du = \int \left(\frac{1}{u} - \frac{1}{u^2}\right)\,du = \ln|u| + \frac{1}{u} + c = \ln|x+1| + \frac{1}{x+1} + c$$

(iv) $\displaystyle\int_1^5 x\sqrt{x-1}\,dx$

Let $u = x - 1$ so $du = dx$ and the limits

$$x = 1 \;\Rightarrow\; u = 0 \quad \text{and} \quad x = 5 \;\Rightarrow\; u = 4$$

and the integral becomes

$$\int_0^4 (u+1)\sqrt{u}\,du = \int_0^4 u^{3/2} + u^{1/2}\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Big|_0^4 = \frac{64}{5} + \frac{16}{3} = \frac{272}{15}$$

(v) $\displaystyle\int_0^{\pi/4} \sin x \cos x\,dx$

Let $u = \sin x$ so $du = \cos x\,dx$ and the limits

$$x = 0 \;\Rightarrow\; u = 0 \quad \text{and} \quad x = \pi/4 \;\Rightarrow\; u = \sqrt{2}/2$$

and the integral becomes

$$\int_0^{\sqrt{2}/2} u\,du = \frac{u^2}{2}\Big|_0^{\sqrt{2/2}} = \frac{1}{4}$$

(vi) $\displaystyle\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$

Let $u = x^2 + 16$ so $du = 2xdx$ and the limits

$$x = 0 \;\Rightarrow\; u = 16 \quad \text{and} \quad x = 3 \;\Rightarrow\; u = 25$$

and the integral becomes

$$\int_{16}^{25} \frac{\frac{1}{2}\,du}{\sqrt{u}} = \sqrt{u}\,\Big|_{16}^{25} = \sqrt{25} - \sqrt{16} = 5 - 4 = 1.$$