Math 6345 - AODE's

So we know

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

and tried to solve

$$\dot{\bar{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x}$$

using this - and it was complicated

However instead we used the fact that

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so $$e^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t} = e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t} \cdot e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t}$$

So can we always do this

Consider the following example

$$e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t} = e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}t} \cdot e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}t}$$

1st $\quad e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t}$

Now $A = \begin{pmatrix}1&1\\0&0\end{pmatrix}$ so $A^2 = \begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix}$

so all the $A^n = \begin{pmatrix}1&1\\0&0\end{pmatrix}$

$$e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\0&0\end{pmatrix}t + \begin{pmatrix}1&1\\0&0\end{pmatrix}\frac{t^2}{2!} + \cdots$$

$$= \begin{pmatrix}1+t+\frac{t^2}{2!}+\frac{t^3}{3!} & t+\frac{t^2}{2!}+\frac{t^3}{3!}\cdots \\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}e^t & e^t-1\\ 0 & 1\end{pmatrix}$$

Next

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so $B^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\text{so } e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{t^2}{2!} \cdots$$

$$= \begin{pmatrix} 1+t+\frac{t^2}{2!}\cdots & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$$

Next $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad C^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so $C^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

and

$$e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}t} \cdot e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}t}$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & te^t \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix}$$

So under what condition is

$$e^{(A+B)t} = e^{At} \cdot e^{Bt}$$

The condition is that $AB = BA$.

<u>Proof</u>

1^{st} $e^{(A+B)t} = I + (A+B)t + \frac{(A+B)^2 t^2}{2!} + \frac{(A+B)^3 t^3}{3!} + \cdots$

Now consider e^{At} & e^{Bt}

so $$e^{At} \cdot e^{Bt} = \left(I + At + \frac{A^2t^2}{2!} + \dots\right)\left(I + Bt + \frac{B^2}{2!}t^2 \dots\right)$$

$$= I + Bt + \frac{B^2}{2!}t^2 + \frac{B^3}{3!}t^3$$

$$+ At + ABt^2 + \frac{AB^2}{2!}t^3 + \dots$$

$$+ \frac{A^2t^2}{2!} + \frac{A^2B}{2!}t^3$$

$$+ \frac{A^3t^3}{3!} + \dots$$

Now compare coeff's involve t^n

$\underline{t}$: $(A+B)t = (B+A)t$

$\underline{t^2}$: $$\frac{(A+B)^2}{2!} = \frac{B^2}{2!} + AB + \frac{A^2}{2!}$$

so $(A+B)(A+B) = B^2 + 2AB + A^2$

$A^2 + AB + BA + B^2 = B^2 + 2AB + A^2$

$\Rightarrow \quad BA = AB \quad$ (A & B commute)

the rest follows by induction.