Math 3331 - Sample Test 3 solns

1. Solve the following using the variation of parameters

$$\begin{aligned} (i) \qquad & y'' + y = \tan x, \\ (ii) \qquad & y'' + 3y' + 2y = \frac{1}{e^x + 1}. \end{aligned}$$

1(i). The homogeneous equation is

$$y'' + y = 0$$

The characteristic equation for this is $m^2 + 1 = 0$ giving $m = \pm i$. Thus, the complementary solution is

$$y = c_1 \sin x + c_2 \cos x.$$

Now we vary the parameters

$$y = u \sin x + v \cos x. \tag{1}$$

Taking the first derivative, we obtain

$$y' = u' \sin x + u \cos x + v' \cos x - v \sin x,$$

from which we set

$$u' \sin x + v' \cos x = 0,$$

leaving

$$y' = u \cos x - v \sin x. \tag{2}$$

Calculating one more derivative gives

$$y'' = u' \cos x - u \sin x - v' \sin x - v \cos x. \tag{3}$$

Substituting (1)and (3) into the original ODE and canceling gives

$$u' \cos x - \cancel{u \sin x} - v' \sin x - \cancel{v \cos x}$$
$$+ \cancel{u \sin x} \qquad\qquad + \cancel{v \cos x} = \tan x$$

or

$$u' \cos x - v' \sin x = \tan x. \tag{4}$$

Equations (2) and (4) are two equations for u' and v' which we solve giving

$$u' = \sin x, \quad v' = -\frac{\sin^2 x}{\cos x}.$$

Integrating each respectively gives

$$u = -\cos x, \quad v = \sin x - \ln|\sec x + \tan x|$$

and from (2) we obtain the particular solution

$$\begin{aligned} y &= -\cos x \sin x + (\sin x - \ln|\sec x + \tan x|)\cos x \\ &= -\cos x \ln|\sec x + \tan x|. \end{aligned} \tag{5}$$

This then gives rise to the general solution

$$y = c_1 \sin x + c_2 \cos x - \cos x \ln|\sec x + \tan x|.$$

1(ii). The homogeneous equation is

$$y'' + 3y' + 2y = 0$$

The characteristic equation for this is $m^2 + 3m + 2 = 0$ giving $m = -1, -2$. Thus, the complementary solution is

$$y = c_1 e^{-x} + c_2 e^{-2x}.$$

Now we vary the parameters

$$y = ue^{-x} + ve^{-2x}. \tag{6}$$

Taking the first derivative, we obtain

$$y' = u'e^{-x} - ue^{-x} + v'e^{-2x} - 2ve^{-2x},$$

from which we set

$$u'e^{-x} + v'e^{-2x} = 0, \tag{7}$$

leaving

$$y' = -ue^{-x} - 2ve^{-2x}. \tag{8}$$

Calculating one more derivative gives

$$y'' = -u'e^{-x} + ue^{-x} - 2v'e^{-2x} + 4ve^{-2x}. \tag{9}$$

Substituting (6), (8) and (9) into the original ODE and canceling gives

$$\begin{aligned} -u'e^{-x} + \cancel{ue^{-x}} - 2v'e^{-2x} &+ \cancel{4ve^{-2x}} \\ - \cancel{3ue^{-x}} \qquad\qquad &- \cancel{6ve^{-2x}} \\ + \cancel{2ue^{-x}} \qquad\qquad &+ \cancel{2ve^{-2x}} = \frac{1}{e^x + 1} \end{aligned} \tag{10}$$

or

$$-u'e^{-x} - 2v'e^{-2x} = \frac{1}{e^x + 1}. \tag{11}$$

Equations (7) and (11) are two equations for u' and v' which we solve giving

$$u' = \frac{e^x}{e^x + 1}, \quad v' = -\frac{e^{2x}}{e^x + 1}.$$

Integrating each respectively gives

$$u = \ln\left(e^x + 1\right), \quad v = -e^x + \ln\left(e^x + 1\right)$$

and from (6) we obtain the particular solution

$$\begin{aligned} y &= \ln\left(e^x + 1\right) e^{-x} + \left(-e^x + \ln\left(e^x + 1\right)\right) e^{-2x} \\ &= \left(e^{-x} + e^{-2x}\right) \ln\left(e^x + 1\right). \end{aligned} \tag{12}$$

noting that the piece e^{-x} can be absorbed into the complementary solution. This then gives rise to the general solution

$$y = c_1 e^{-x} + c_2 e^{-2x} + \left(e^{-x} + e^{-2x}\right) \ln\left(e^x + 1\right).$$

2(i)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix} \bar{x} \tag{13}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda - 1 & -2 \\ -5 & \lambda + 2 \end{vmatrix} = \lambda^2 + \lambda - 12 = (\lambda + 4)(\lambda - 3) = 0,$$

from which we obtain the eigenvalues $\lambda = -4$ and $\lambda = 3$.

Case 1: $\lambda = -4$

In this case we have

$$\begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $5c_1 + 2c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 2 \\ -5 \end{pmatrix},$$

so one solution is

$$\bar{x}_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-4t}.$$

Case 2: $\lambda = 3$

In this case we have

$$\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

from which we obtain the other solution

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

The general solution to (13) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

2(ii)

Consider

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \bar{x}, \quad \bar{x}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{14}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-3 \end{vmatrix} = \lambda^2 - 4\lambda + 9 = (\lambda-2)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 2$ and $\lambda = 2$ − repeated. As in problem 2(i) we find the eigenvector associated with this

Case 1: $\lambda = 2$

In this case we have

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so one solution is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

For the second independent solution we seek a second solution of the form

$$\bar{x}_2 = \bar{u}te^{2t} + \bar{v}e^{2t}. \tag{15}$$

As shown in class, $\bar{u} = \bar{c}$ and $\vec{v}$ satisfies

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{16}$$

or $v_1 + v_2 = -1$. Here, we'll choose

$$\bar{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Therefore, the second solution is

$$\bar{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t}$$

and the general solution

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} \right],$$

Imposing the initial condition gives

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

This gives $c_1 - c_2 = 5$ and $-c_1 = -2$ so $c_1 = 2$ and $c_2 = -3$. The general solution then becomes

$$\bar{x} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} - 3 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} \right],$$

2(iii)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \bar{x}. \tag{17}$$

The characteristic equation is

$$\begin{vmatrix} \lambda - 6 & 1 \\ -5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

Using the quadratic formula, we obtain $\lambda = 5 \pm 2i$ (so $\alpha = 5$ and $\beta = 2$). For the eigenvectors, we wish to solve

$$\begin{pmatrix} 5+2i-6 & 1 \\ -5 & 5+2i-4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{pmatrix} -1+2i & 1 \\ -5 & 1+2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means solving

$$-5v_1 + (1+2i)v_2 = 0.$$

One solution is

$$\bar{v} = \begin{pmatrix} 1+2i \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i.$$

So here

$$\bar{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \vec{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

With $\alpha = 5$ and $\beta = 2$ gives

$$\vec{x}_1 = \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t}, \quad \vec{x}_2 = \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}.$$

The general solution is just a linear combination of these two

$$\vec{x} = c_1 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t} + c_2 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}.$$