

# Math 4315 - PDEs

## Ordinary Differential Equations Review

### 1 First Order Equations

Ordinary differential equations of the form

$$y' = F(x, y) \quad (1)$$

are called first order ordinary differential equations. There are a variety of techniques to solve these type of equations and main methods are:

- (i) separable
- (ii) linear
- (iii) Bernoulli
- (iv) Ricatti
- (v) homogeneous
- (vi) linear fractional
- (vii) exact

#### 1.1 Separable Equations

A separable first order differential equation has the form:

$$\frac{dy}{dx} = f(x)g(y) \quad (2)$$

The general solution is found by separating the differential equation and integrating including a single constant of integration, *i.e.*

$$\int \frac{1}{g(y)} dy = \int f(x) dx + c.$$

For example, to solve

$$y' = 2xy - 6x + 3y - 9,$$

it is necessary to rewrite it as

$$y' = (2x + 3)(y - 3).$$

Separating variables and writing in integral form gives

$$\int \frac{dy}{y-3} = \int 2x + 6 dx,$$

and integrating yields

$$\ln |y - 3| = x^2 + 3x + c.$$

Letting  $c = \ln(k)$  and solving for  $y$  gives:

$$y = k \cdot e^{x^2+3x} + 3.$$

## 1.2 Linear Equations

Equations of this type are in the form

$$\frac{dy}{dx} + p(x)y = q(x) \quad (3)$$

To solve this, we introduce the integrating factor

$$\mu = e^{\int p(x)dx} \quad (4)$$

This is designed so that when both sides of (3) are multiplied by  $\mu$ , the left side (3) is a derivative of a product, that is, it becomes:

$$\mu \left( \frac{dy}{dx} + py \right) = \frac{d}{dx}(\mu y),$$

and then (3) can be integrated. For example, if

$$\frac{dy}{dx} - \frac{2y}{x} = 2x^3 - 1, \quad (5)$$

then  $p(x) = -\frac{2}{x}$  so that

$$\mu = e^{-2 \int \frac{dx}{x}} = e^{-2 \ln x} = \frac{1}{x^2}.$$

On multiplying (5) by  $\mu$  gives

$$\frac{1}{x^2} \frac{dy}{dx} - \frac{2y}{x^3} = 2x - \frac{1}{x^2},$$

which simplifies to

$$\frac{d}{dx} \left( \frac{1}{x^2} \cdot y \right) = 2x - \frac{1}{x^2}.$$

Integrating gives

$$\frac{1}{x^2} \cdot y = x^2 + \frac{1}{x} + c,$$

and solving for  $y$  gives

$$y = x^4 + x + cx^2.$$

### 1.3 Bernoulli

Equations of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad (n \neq 0, 1) \quad (6)$$

are called Bernoulli equations. Dividing both sides of (6) by  $y^n$  gives

$$\frac{1}{y^n} \frac{dy}{dx} + \frac{p(x)}{y^{n-1}} = q(x) \quad (7)$$

Let  $v = \frac{1}{y^{n-1}}$ , then  $\frac{dv}{dx} = (1-n)\frac{1}{y^n} \frac{dy}{dx}$  or  $\frac{1}{(1-n)} \frac{dv}{dx} = \frac{1}{y^n} \frac{dy}{dx}$ . Upon making this substitution into (7) gives

$$\frac{1}{1-n} \frac{dv}{dx} + p(x)v = q(x)$$

which is linear. So Bernoulli equations can be reduced to linear equations.

Example:

$$\frac{dy}{dx} - \frac{y}{2x} = y^3.$$

This is an example where  $n = 3$ . Putting this into standard form gives

$$\frac{1}{y^3} \frac{dy}{dx} - \frac{1}{2x} \frac{1}{y^2} = 1 \quad (8)$$

Letting  $v = \frac{1}{y^2}$ , then  $\frac{dv}{dx} = \frac{-2}{y^3} \frac{dy}{dx}$  and (8) is transformed to

$$-\frac{1}{2} \frac{dv}{dx} - \frac{1}{2x} v = 1,$$

or

$$\frac{dv}{dx} + \frac{v}{x} = -2.$$

As this is linear, then  $p(x) = \frac{1}{x}$ , and the integrating factor for this is  $x$ , so that

$$x \frac{dv}{dx} + v = \frac{d}{dx} (xv) = -2x,$$

and thus

$$xv = -x^2 + c,$$

or

$$v = -x + \frac{c}{x}.$$

So that

$$\frac{1}{y^2} = -x + \frac{c}{x},$$

or

$$y = \frac{\pm 1}{\sqrt{-x + \frac{c}{x}}}.$$

## 1.4 Riccati Equations

Riccati equations have the form:

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x). \quad (9)$$

To find a general solution to this requires having one solution first. Given this solution, it is possible to change equation (9) to a linear equation. If we let

$$y = y_0 + \frac{1}{u},$$

where  $y_0$  is a solution to (9), then (9) is transformed to the linear equation

$$u' = -(2a(x)y_0 + b(x))u - a(x),$$

which is linear. To illustrate, we consider the following example

$$\frac{dy}{dx} = -\frac{y^2}{x^2} + \frac{y}{x} + 1. \quad (10)$$

Since  $y_0 = x$  is a solution to this equation, let  $y = x + \frac{1}{u}$ , and (10) becomes

$$1 - \frac{u'}{u^2} = -\frac{1}{x^2} \left( x^2 + 2\frac{x}{u} + \frac{1}{u^2} \right) + \frac{1}{x} \left( x + \frac{1}{u} \right) + 1.$$

Simplifying gives

$$-\frac{u'}{u^2} = -\frac{1}{xu} - \frac{1}{x^2u^2},$$

and multiplying by  $-u^2$  and rearranging gives rise to the linear equation

$$u' - \frac{1}{x}u = \frac{1}{x^2}. \quad (11)$$

Here  $p(x) = -\frac{1}{x}$  so this has the integrating factor  $\mu = e^{\int \frac{-dx}{x}} = \frac{1}{x}$ , so (11) becomes

$$\frac{u'}{x} - \frac{u}{x^2} = \frac{d}{dx} \left( \frac{u}{x} \right) = \frac{1}{x^3},$$

and upon integration gives

$$\frac{u}{x} = -\frac{1}{2x^2} + c,$$

or

$$u = cx - \frac{1}{2x}.$$

Since  $y = x + \frac{1}{u}$ , this gives  $y$  as

$$y = x + \frac{1}{cx - \frac{1}{2x}}.$$

## 1.5 Homogeneous Equations

Equations of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad (12)$$

are called homogeneous equations. Substituting  $y = xu$  will yield the equation

$$x \frac{du}{dx} + u = F(u).$$

which separates to

$$\frac{du}{F(u) - u} = \frac{dx}{x}.$$

Consider the previous example,

$$\frac{dy}{dx} = -\frac{y^2}{x^2} + \frac{y}{x} + 1. \quad (13)$$

If we let  $y = xu$ , then (13) becomes

$$\frac{d(xu)}{dx} = -u^2 + u + 1,$$

or

$$x \frac{du}{dx} + u = -u^2 + u + 1$$

and simplifying and separating gives

$$\frac{du}{1 - u^2} = \frac{dx}{x}.$$

Integrating gives

$$\frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| = \ln |x| + \ln c,$$

or

$$\frac{u+1}{u-1} = cx^2$$

Since  $y = xu$  this gives

$$\frac{\frac{y}{x} + 1}{\frac{y}{x} - 1} = cx^2$$

or

$$\frac{y+x}{y-x} = cx^2$$

solving for  $y$  leads to the solution

$$y = \frac{cx^3 + x}{cx^2 - 1}.$$

## 1.6 Linear Fractional

Equations that have the form

$$\frac{dy}{dx} = \frac{ax + by + e}{cx + dy + f}, \quad (14)$$

are called linear fractional. Under a change of variables,

$$x = \bar{x} + \alpha, \quad y = \bar{y} + \beta$$

we can change equation (14) to one that is either homogeneous ( $ad - bc \neq 0$ ) or to one that is separable ( $ad - bc = 0$ ). The following examples illustrate. Consider

$$\frac{dy}{dx} = \frac{2x - 3y + 8}{3x - 2y + 7}. \quad (15)$$

If we let

$$x = \bar{x} + \alpha, \quad y = \bar{y} + \beta,$$

then (15) becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{x} - 3\bar{y} + 2\alpha - 3\beta + 8}{3\bar{x} - 2\bar{y} + 3\alpha - 2\beta + 7}.$$

Choosing

$$2\alpha - 3\beta + 8 = 0, \quad 3\alpha - 2\beta + 7 = 0, \quad (16)$$

leads to

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{x} - 3\bar{y}}{3\bar{x} - 2\bar{y}}, \quad (17)$$

a homogeneous equation. The natural question is, “does (16) have a solution?” In this case, it does and we can find that the solution is  $\alpha = -1$  and  $\beta = 2$ . The solution of (17) is

$$\bar{x}^2 - 3\bar{x}\bar{y} + \bar{y}^2 = c \quad (18)$$

and from our change of variables ( $x = \bar{x} - 1$ ,  $y = \bar{y} + 2$ ) we find the solution to (15) is

$$(x + 1)^2 - 3(x + 1)(y - 2) + (y - 2)^2 = c. \quad (19)$$

As a second example, consider

$$\frac{dy}{dx} = \frac{4x - 2y + 8}{2x - y + 7}. \quad (20)$$

If we let

$$x = \bar{x} + \alpha, \quad y = \bar{y} + \beta,$$

then

$$\frac{d\bar{y}}{d\bar{x}} = \frac{4\bar{x} - 2\bar{y} + 4\alpha - 2\beta + 8}{2\bar{x} - \bar{y} + 2\alpha - \beta + 7}.$$

We choose

$$4\alpha - 2\beta + 8 = 0, \quad 2\alpha - \beta + 7 = 0,$$

but this has no solution! However, if we let  $u = 2x - y$ , then (20) becomes

$$\frac{du}{dx} = \frac{6}{u + 7},$$

which is separable! Its solution is given by

$$u^2 + 14u = 12x + c,$$

and upon back substitution, we obtain the solution of (20) as

$$(2x - y)^2 + 14(2x - y) = 12x + c,$$

## 1.7 Exact Equations

An ordinary differential equation of the form

$$\frac{dy}{dx} = F(x, y), \quad (21)$$

has the alternate form

$$M(x, y)dx + N(x, y)dy = 0. \quad (22)$$

If  $M$  and  $N$  have continuous partial derivatives of first order in some region  $R$  and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the ODE (21) or (22) is said to be “exact” and can be integrated by setting

$$\frac{\partial \phi}{\partial x} = M, \text{ and } \frac{\partial \phi}{\partial y} = N.$$

For example, consider the differential equation

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + y^2}, \quad (23)$$

which can be written as

$$2xy \, dx + (x^2 + y^2) \, dy = 0. \quad (24)$$

If we identify that  $M$  and  $N$  are

$$M = 2xy \text{ and } N = x^2 + y^2,$$

so

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x,$$

so  $2xy \, dx + (x^2 + y^2) \, dy = 0$  is exact. Setting

$$\frac{\partial \phi}{\partial x} = 2xy, \text{ and } \frac{\partial \phi}{\partial y} = x^2 + y^2.$$

then from the first of these equations upon integrating we get

$$\phi = x^2y + g(y),$$

taking the partial of this with respect to  $y$  gives

$$\frac{\partial \phi}{\partial y} = x^2 + g'(y).$$

Comparing this to  $\frac{\partial \phi}{\partial y} = x^2 + y^2$  gives that

$$g'(y) = y^2,$$

so

$$g(y) = \frac{y^3}{3} + c,$$



so that

$$\phi = x^2y + \frac{y^3}{3} + c.$$

From Cal III we know that

$$d\phi = \phi_x dx + \phi_y dy,$$

but in this case this is

$$d\phi = 2xy dx + (x^2 + y^2) dy = 0$$

so  $\phi$  is a constant. Thus we have as solutions to

$$2xy dx + (x^2 + y^2) dy = 0$$

$$\phi = k \text{ or } x^2y + \frac{y^3}{3} + c = k,$$

and absorbing the constant  $k$  into  $c$  gives

$$x^2y + \frac{y^3}{3} + c = 0$$

as the set of possible solutions to (23).