Math 4381/6357

Nonlinear Partial Differential Equations

Charpit's Method

Obtaining exact solutions to nonlinear PDEs such as

$$xu_x - u_y^2 = 2u \tag{1}$$

is quite difficult as we are required to solve equations such as

$$x_s = x, \tag{2a}$$
$$y_s = -2q, \tag{2b}$$
$$u_s = xp - 2q^2, \tag{2c}$$
$$p_s = p, \tag{2d}$$
$$q_s = 2q. \tag{2e}$$

It's not so much in solving these characteristic equations but eliminating the 5 arbitrary functions that appear upon integration. So we ask, is it possible to come up with exact solutions another way?

Consider the PDE

$$u_y = -y. \tag{3}$$

This integrates to give

$$u = -\frac{y^2}{2} + f(x) \tag{4}$$

and substitution into the original PDE (1) gives

$$xf' = 2f. \tag{5}$$

This ODE is solved giving

$$f = c_1 x^2 \tag{6}$$

which from (4) leads to

$$u = c_1 x^2 - \frac{y^2}{2}. \tag{7}$$

Consider the PDE

$$u_x = x. \tag{8}$$

This integrates to give

$$u = \frac{x^2}{2} + g(y) \tag{9}$$

and substitution into the original PDE (1) gives

$$-g'^2 = 2g. \tag{10}$$

This ODE is solved giving (we will omit the trivial solution $g = 0$)

$$g = -\frac{y^2}{2} + c_2 y - \frac{c_2^2}{2}. \tag{11}$$

leading to the solution

$$u = \frac{x^2}{2} - \frac{y^2}{2} + c_2 y - \frac{c_2^2}{2} \tag{12}$$

noting that setting

As for the final example, consider the PDE

$$u_x + x u_y = x - xy. \tag{13}$$

This integrates to give

$$u = -\frac{1}{2}(y-1)^2 + f\left(y - \frac{1}{2}x^2\right) \tag{14}$$

and substitution into the original PDE (1) gives

$$2(\lambda - 1)f' - f'^2 = 2f \tag{15}$$

where $f = f(\lambda)$ and $\lambda = y - \frac{1}{2}x^2$. This ODE actually has two solutions

$$f = c\lambda - c_3 - \frac{1}{2}c_3^2, \quad f = \frac{(\lambda-1)^2}{2}, \tag{16}$$

and lead to the exact solutions

$$\begin{aligned} u &= -\frac{1}{2}(y-1)^2 + c_3\left(y - \frac{1}{2}x^2\right) - c_3 - \frac{1}{2}c_3^2, \\ u &= -\frac{1}{2}(y-1)^2 + \frac{1}{2}\left(y - \frac{1}{2}x^2 - 1\right)^2 > \end{aligned} \tag{17}$$

So are there others? For example, both

$$\begin{aligned} u_y &= x^2 \\ x u_x + 2y u_y &= 2u - y^2 \end{aligned} \tag{18}$$

will lead to exact solutions of the given PDE. In fact any PDE of the form

$$F\left(\frac{u_x}{x}, \frac{u_x^2}{u_y}, u_y + y, xu_x - u_y^2 - 2u\right) = 0 \tag{19}$$

will give rise to exact solution to (1). A number of questions arise.

1. Where did these associated PDEs come from?

2. How do we know that they will lead to a solution that also satisfies the BC?

Before trying to answer such questions, it is important to know that an actual solution exists. Namely, does a solution exist that satisfies both the original PDE and second one that we appended to the original. So, in the first example, does a solution exist that satisfies both

$$xu_x - u_y^2 = 2u \text{ and } u_y = -y? \tag{20}$$

Here, we use the second in the first and ask, does there exist an solution to

$$u_x = \frac{2u}{x} + \frac{y^2}{x}, \quad u_y = -y? \tag{21}$$

If so, then they certainly would be compatible so $\dfrac{\partial u_x}{\partial y} = \dfrac{\partial u_y}{\partial x}$. Calculating these gives

$$\frac{2u_y}{x} + \frac{2y}{x} \stackrel{?}{=} 0 \tag{22}$$

and since $u_y = -y$ this gives

$$-\frac{2y}{x} + \frac{2y}{x} \stackrel{?}{=} 0 \tag{23}$$

which is true, so the two equations are compatible. For the second example we ask, are the following compatible

$$xu_x - u_y^2 = 2u \text{ and } u_x = x? \tag{24}$$

We certainly could substitute the second into the first and solve for u_y and then seek compatibility but instead we consider

$$x^2 - u_y^2 = 2u \text{ and } u_x = x? \tag{25}$$

Now we differentiate the first with respect to x giving

$$2x - 2u_y u_{xy} = 2u_x \tag{26}$$

and since $u_x = x$ then (26) is identically satisfied. For the final example, we ask are these compatible?

$$xu_x - u_y^2 = 2u \text{ and } u_x + xu_y = x - xy? \tag{27}$$

Definitely a harder problem to explicitly find u_x and u_y but is that really necessary? If we calculate the x and y derivatives of each we obtain

$$xu_{xx} - 2u_y u_{xy} = u_x, \tag{28a}$$

$$xu_{xy} - 2u_y u_{yy} = 2u_y, \tag{28b}$$

$$u_{xx} + xu_{xy} + u_y = 1 - y, \tag{28c}$$

$$u_{xy} + xu_{yy} = -x. \tag{28d}$$

Eliminating u_{xy} from (28b) and (28d)

$$(2u_y + x^2)(u_{yy} + 1) = 0 \tag{29}$$

so we see two cases emerge:

$$\text{(i)} \quad 2u_y + x^2 = 0,$$

$$\text{(ii)} \quad 2u_y + x^2 \neq 0.$$

In the first case $u_y = -\frac{1}{2}x^2$, (28) reduces to

$$2u_y + x^2 = 0, \quad u_x - \frac{1}{2}x^3 - x + xy = 0,$$

and these are compatible whereas in the second case we obtain $u_{yy} = -1$, (28) reduces to

$$u_x + xu_y + x(y-1) = 0$$

which is identically satisfied by virtue of (27).

So now we know how to determine when two PDEs are compatible. Our next step is to determine how they come about.

Consider the compatibility of the following first order PDEs

$$\begin{aligned} F(x, y, u, p, q) &= 0, \\ G(x, y, u, p, q) &= 0, \end{aligned} \tag{30}$$

where $p = u_x$ and $q = u_y$. Calculating the x and y derivatives of (30) gives

$$\begin{aligned} F_x + pF_u + u_{xx}F_p + u_{xy}F_q &= 0, \\ F_y + qF_u + u_{xy}F_p + u_{yy}F_q &= 0, \\ G_x + pG_u + u_{xx}G_p + u_{xy}G_q &= 0, \\ G_y + qG_u + u_{xy}G_p + u_{yy}G_q &= 0. \end{aligned} \tag{31}$$

Solving the first three (31) for u_{xx}, u_{xy} and u_{yy} gives

$$\begin{aligned}
u_{xx} &= \frac{-F_x\ G_q - p\ F_u\ G_q + F_q\ G_x + p\ F_q\ G_u}{F_p\ G_q - F_q\ G_p}, \\
u_{xy} &= \frac{-F_p\ G_x - p\ F_p\ G_u + F_x\ G_p + p\ F_u\ G_p}{F_p\ G_q - F_q\ G_p}, \\
u_{yy} &= \frac{\begin{array}{c} F_p^2\ G_x + p\ F_p^2\ G_u - F_y\ F_p\ G_q - q\ F_u\ F_p\ G_q \\ +q\ F_u\ F_q\ G_p - F_x\ F_p\ G_p - p\ F_u\ F_p\ G_p + F_y\ F_q\ G_p \end{array}}{(F_p\ G_q - F_q\ G_p)F_q}.
\end{aligned}$$

Substitution into the last of (31) gives

$$F_p\ G_x + F_q\ G_y + (p\ F_p + q\ F_q)G_u - (F_x + p\ F_u)G_p - (F_y + q\ F_u)G_q = 0,$$

or conveniently written as

$$\begin{vmatrix} D_xF & F_p \\ D_xG & G_p \end{vmatrix} + \begin{vmatrix} D_yF & F_q \\ D_yG & G_q \end{vmatrix} = 0, \tag{33}$$

where $D_x\ F = F_x + p\ F_u$, $D_y\ F = F_y + q\ F_u$ and $|\ \cdot\ |$ the usual determinant. These are known as the *Charpit equations*. We also assumed that $F_pG_q - F_qG_p \neq 0$ and $F_q \neq 0$. These cases would need to be considered separately.

Example 1.1 Consider

$$xu_x - u_y^2 = 2u \tag{34}$$

This is the example we considered already, however now we will determine all classes of equation that are compatible with this one. Denoting

$$\begin{aligned}
G &= xu_x - u_y^2 - 2u \\
&= xp - q^2 - 2u
\end{aligned}$$

where $p = u_x$ and $q = u_y$, then

$$G_x = p,\ \ G_y = 0,\ \ G_u = -2,\ \ G_p = x,\ \ G_q = -2q,$$

and the Charpit equations are

$$\begin{vmatrix} D_xF & F_p \\ -p & x \end{vmatrix} + \begin{vmatrix} D_yF & F_q \\ -2q & -2q \end{vmatrix} = 0,$$

or, after expansion

$$xF_x - 2qF_y + \left(xp - 2q^2\right) F_u + pF_p + 2qF_q = 0.$$

Solving this linear PDE by the method of characteristics gives the solution as

$$F\left(\frac{p}{x}, \frac{p^2}{q}, q + y, xp - q^2 - 2u\right) = 0 \tag{35}$$

which is exactly the one given in (19)!

Example 1.2 Consider

$$u_x u_y = 1 \tag{36}$$

Denoting

$$G = u_x u_y - 1 = pq - 1,$$

where $p = u_x$ and $q = u_y$, then

$$G_x = 0, \;\; G_y = 0, \;\; G_u = 0, \;\; G_p = q, \;\; G_q = p,$$

and the Charpit equations are

$$\begin{vmatrix} D_x F & F_p \\ 0 & q \end{vmatrix} + \begin{vmatrix} D_y F & F_q \\ 0 & -p \end{vmatrix} = 0,$$

or, after expansion

$$qF_x + pF_y + 2pqF_u = 0,$$

noting that the third term can be replaced by $2F_u$ due to the original equation. Solving this linear PDE by the method of characteristics gives the solution as

$$F = F(uq - 2x, up - 2y, p, q). \tag{37}$$

or

$$F = F(uu_y - 2x, uu_x - 2y, \; u_x, \; u_y). \tag{38}$$

For example, if we choose

$$uu_y - 2x = 0 \tag{39}$$

then on integrating we obtain

$$u^2 = 4xy + f(x) \tag{40}$$

and substituting into the original PDE gives

$$xf' = f \tag{41}$$

and leads to the exact solution

$$u^2 = 4xy + cx. \tag{42}$$

If we choose

$$uu_x + u_y - 2y = 0 \tag{43}$$

then on integrating we obtain

$$u - y^2 + f\left(x + \frac{2}{3}y^3 - yu\right) = 0 \tag{44}$$

and substituting into the original PDE gives

$$ff'^2 - 1 = 0 \tag{45}$$

and leads to the exact solution

$$u - y^2 \pm \left(\frac{3}{2}(x + \frac{2}{3}y^3 - yu + c)\right)^{2/3} = 0. \tag{46}$$

Example 1.3 Consider

$$u_x^2 + u_y^2 = u^2. \tag{47}$$

Denoting $p = u_x$ and $q = u_y$, then

$$G = u_x^2 + u_y^2 - u^2 = p^2 + q^2 - u^2.$$

Thus

$$G_x = 0,\ \ G_y = 0,\ \ G_u = -2u,\ \ G_p = 2p,\ \ G_q = 2q,$$

and the Charpit equation's are

$$\begin{vmatrix} D_xF & F_p \\ -2pu & 2p \end{vmatrix} + \begin{vmatrix} D_yF & F_q \\ -2qu & 2q \end{vmatrix} = 0,$$

or, after expansion

$$pF_x + qF_y + \left(p^2 + q^2\right) F_u + puF_p + quF_q = 0, \tag{48}$$

noting that the third term can be replaced by u^2F_u due to the original equation. Solving (48), a linear PDE, by the method of characteristics gives the solution as

$$F = F\left(x - \frac{p}{u}\ln u, y - \frac{q}{u}\ln u, \frac{p}{u}, \frac{q}{u}\right).$$

Consider the following particular example

$$x - \frac{p}{u}\ln u + y - \frac{q}{u}\ln u = 0,$$

or

$$u_x + u_y = (x+y)\frac{u}{\ln u}.$$

If we let $u = \mathrm{e}^{\sqrt{v}}$ then this becomes

$$v_x + v_y = 2(x+y),$$

which, by the method of characteristics, has the solution

$$v = 2xy + f(x-y).$$

This, in turn, gives the solution for u as

$$u = \mathrm{e}^{\sqrt{2xy+f(x-y)}}. \tag{49}$$

Substitution into the original equation (47) gives the following ODE

$$f'^2 - 2\lambda f' - 2f + 2\lambda^2 = 0,$$

where $f = f(\lambda)$ and $\lambda = x - y$. If we let $f = g + \frac{1}{2}\lambda^2$ then we obtain

$$g'^2 - 2g = 0 \tag{50}$$

whose solution is given by

$$g = \frac{(\lambda + c)^2}{2}, \quad g = 0 \tag{51}$$

where c is an arbitrary constant of integration. This, in turn, gives

$$f = \lambda^2 + c\lambda + \frac{1}{2}c^2, \quad f = \frac{1}{2}\lambda^2 \tag{52}$$

and substitution into (49) gives

$$u = \mathrm{e}^{\sqrt{x^2+y^2+c(x-y)+\frac{1}{2}c^2}}, \quad u = \mathrm{e}^{\sqrt{2xy+(x-y)^2/2}},$$

as an exact solution to the original PDE.

It is interesting to note that when we substitute the solution of the compatible equation into the original it reduces to an ODE. A natural question is, does this always happen? This was proven to be true in two independent variables.
Daniel J. Arrigo, Nonclassical Contact Symmetries and Charpit's Method of Compatibility, *J. Non Math Phys.* **12**(3), 321-329 (2005).

Boundary Conditions

So now we ask, of the infinite possibilities here, can we choose the right one(s) to give rise to the solution that also satisfies the given boundary condition? The following example illustrates.

Example 1.4 Solve

$$u_x u_y - x u_x - y u_y = 0 \tag{53}$$

subject to the boundary conditions

$$\begin{aligned} (i)\ \ & u(x,0) = 0 \\ (ii)\ \ & u(x,0) = \frac{1}{2}x^2 \\ (iii)\ \ & u(x,x) = 2x^2 \end{aligned} \tag{54}$$

Denoting

$$G = pq - xp - yq,$$

where $p = u_x$ and $q = u_y$, then

$$G_x = -p,\ \ G_y = -q,\ \ G_u = 0,\ \ G_p = q - x,\ \ G_q = p - y,$$

and the Charpit equations are

$$\begin{vmatrix} D_x F & F_p \\ -p & q - x \end{vmatrix} + \begin{vmatrix} D_y F & F_q \\ -q & p - y \end{vmatrix} = 0,$$

or, after expansion

$$(q - x)F_x + (p - y)F_y + (2pq - xp - yq)F_u + pF_p + qF_q = 0,$$

noting that the third term can be replaced by pqF_u due to the original equation. Solving this linear PDE by the method of characteristics gives the solution as

$$F = F(q^2 - 2xq, p^2 - 2yp, p/q, pq - 2u). \tag{55}$$

or

$$F = F\left(u_y^2 - 2xu_y, u_x^2 - 2yu_x, \frac{u_x}{u_y}, u_x u_y - 2u\right). \tag{56}$$

So how do we incorporate the boundary conditions? We will look at each separately.

Boundary Condition (i) In this case $u(x,0) = 0$ and differentiating with respect to x gives $u_x(x,0) = 0$ or $p = 0$ on the boundary. From the original PDE we then have $q = 0$. Substituting these into (62) gives

$$F = F(0, 0, ?, 0), \tag{57}$$

noting that we have included a ? in the third argument of F since where have $\frac{0}{0}$. So what does this mean? If we choose any of the arguments that are zero, it means the boundary conditions are satisfied by that particular PDE. Thus, if we choose the first, for example, we have

$$u_y^2 - 2xu_y = 0 \tag{58}$$

and we know that this is compatible with the original PDE and satisfies the BC, it means that it will give rise to a solution. As there are two cases (i) $u_y = 0$ and (ii) $u_y - 2x = 0$, we consider each separately.

Case (i) If $u_y = 0$ then $u_x = 0$ from the original PDE giving $u = c$ and the BC $u(x,0) = 0$ gives $c = 0$ so the solution is $u \equiv 0$.

Case (ii) In the second case where $u_y - 2x = 0$, integrating gives $u = 2xy + g(x)$ and substituting into the original PDE gives $xf' = 0$ so $f = c$. Thus, we have the exact solution $u = 2xy + c$. The BC $u(x,0) = 0$ gives that $c = 0$ and so the solution is $u = 2xy$.

Boundary Condition (ii) In this case $u(x,0) = \dfrac{1}{2}x^2$ and differentiating with respect to x gives $u_x(x,0) = x$ or $p = x$ on the boundary. From the original PDE we then have $xq - x^2 = 0$ or $q = x$. Substituting these into (62) gives

$$F = F(-x^2, x^2, 1, 0), \tag{59}$$

So what does this mean? Again, if we choose any combination of the arguments that is zero (combined), then the boundary conditions are satisfied by that particular PDE. Thus, if we choose the sum of the first two arguments i.e. $u_y^2 - 2xu_y + u_x^2 - 2yu_x = 0$ then the solution of this will satisfiy the BC $u(x,0) = \frac{1}{2}x^2$. However, this PDE is nonlinear. As we wish to solve a linear problem if we can we will choose different. Another choice is $p/q = 1$. So we are to solve

$$u_x - u_y = 0. \tag{60}$$

This is easily solved giving $u = f(x+y)$ and substitution into the original PDE gives

$$f'^2 - \lambda f' = 0 \quad \text{or} \quad f' = 0, \quad f' - \lambda = 0, \tag{61}$$

where $\lambda = x + y$. Only the second gives rise to a correct solution and we find that $f = \frac{1}{2}\lambda^2$ and leads to the exact solution $u = \dfrac{1}{2}(x+y)^2$

Boundary Condition (iii) In this case $u(x,x) = 2x^2$ and differentiating with respect to x gives $u_x(x,x) + u_y(x,y) = 4x$ or $p + q = 4x$ on the boundary. From the original PDE we then have $pq - xp - xq = 0$. Solving for p and q gives $p = q = 2x$. Substituting these into (62) gives

$$F = F(0,0,1,0), \tag{62}$$

So now we have lots of possibilities Again, if we choose any combination of the arguments that is zero (combined), then the boundary conditions are satisfied by that particular PDE. For example,

$$\begin{aligned}
&(a)\ \ u_y^2 - 2xu_y = 0,\\
&(b)\ \ u_x^2 - 2yu_x = 0,\\
&(c)\ \ u_x/u_y - 1 = 0,\\
&(d)\ \ u_xu_y - 2u = 0,
\end{aligned} \tag{63}$$

will all leads to solutions that satisfy the BC. PDE (a) leads to $u = 2xy$. PDE (b) leads to the same solution. PDE (c) leads to $u = \frac{1}{2}(x+y)^2$ and for the last PDE (d), we use the original PDE (53) so we solve

$$xu_x + yu_u = 2u \tag{64}$$

This is easily solved giving $u = x^2 f(y/x)$ and substitution into the original PDE gives

$$2ff' - \lambda f'^2 - 2f = 0. \tag{65}$$

Now (65) is nonlinear but is homogeneous and can be solved giving the solution is $f = \frac{1}{2c_1}\left(\lambda + c_1\right)^2$ but with the boundary condition ultimately leads to $u = \frac{1}{2}(x+y)^2$.