

## Spring 2026 – Math 3331 Second Order ODEs

In general, second order linear ODEs are of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x). \quad (1)$$

The homogeneous equation ( $f(x) = 0$ ) consists of two linearly independent solutions ( $y_1$  and  $y_2$ ) and the general solution is

$$y = c_1y_1(x) + c_2y_2(x) \quad (2)$$

often called the complementary solution. The general solution of the nonhomogeneous equation ( $f(x) \neq 0$ ) is

$$y = y_h + y_p \quad (3)$$

where  $y_h$  is the solution of the homogeneous equation and  $y_p$  a particular solution of the the entire equation (1).

Example 1. Show  $y_1 = e^x$  and  $y_2 = e^{2x}$  satisfy the ODE

$$y'' - 3y' + 2y = 0. \quad (4)$$

*Sol:*

Here  $y_1 = e^x$ , so  $y_1' = e^x$  and  $y_1'' = e^x$  and substituting gives

$$y'' - 3y' + 2y = e^x - 3e^x + 2e^x = 0 \quad \checkmark \quad (5)$$

For the second solution,  $y_2 = e^{2x}$ , so  $y_2' = 2e^{2x}$  and  $y_2'' = 4e^{2x}$  and substituting gives

$$y'' - 3y' + 2y = 4e^{2x} - 3 \cdot 2e^{2x} + 2e^{2x} = 0 \quad \checkmark \quad (6)$$

The general solution is

$$y_h = c_1e^x + c_2e^{2x} \quad (7)$$

Example 2. Show  $y = c_1e^{-x} + c_2xe^{-x}$  satisfies the ODE

$$y'' + 2y' + y = 0. \quad (8)$$

Sol:

Here

$$\begin{aligned}y &= c_1 e^{-x} + c_2 x e^{-x} \\y' &= -c_1 e^{-x} + c_2 (e^{-x} - x e^{-x}) \\y'' &= c_1 e^{-x} + c_2 (-2e^{-x} + x e^{-x})\end{aligned}\tag{9}$$

and substituting gives

$$\begin{aligned}y'' + 2y' + y &= c_1 e^{-x} + c_2 (-2e^{-x} + x e^{-x}) + 2[-c_1 e^{-x} + c_2 (e^{-x} - x e^{-x})] + c_1 e^{-x} + c_2 x e^{-x} \\&= c_1 (e^{-x} - 2e^{-x} + e^{-x}) \\&\quad + c_2 (-2e^{-x} + x e^{-x} + 2(e^{-x} - x e^{-x}) + x e^{-x}) = 0 \checkmark\end{aligned}\tag{10}$$

Example 3. Show  $y = c_1 \sin x + c_2 \cos x + x^2$  satisfies the ODE

$$y'' + y = x^2 + 2.\tag{11}$$

Sol: Note that the ODE is nonhomogeneous. Here we directly substitute so

$$\begin{aligned}y &= c_1 \sin x + c_2 \cos x + x^2 \\y' &= c_1 \cos x - c_2 \sin x + 2x \\y'' &= -c_1 \sin x - c_2 \cos x + 2\end{aligned}\tag{12}$$

and substituting gives

$$y'' + y = c_1 \sin x + c_2 \cos x + x^2 - c_1 \sin x - c_2 \cos x + 2 = x^2 + 2 \checkmark\tag{13}$$

### Linear Independence

Two functions  $f(x)$  and  $g(x)$  are said to be linearly independent if

$$c_1 f(x) + c_2 g(x) = 0 \quad \text{iff} \quad c_1 = c_2 = 0\tag{14}$$

For example, are  $\{1, x\}$  linearly independent? We consider

$$c_1(1) + c_2 x = 0\tag{15}$$

As this must be true for all  $x$  it is certainly must be true for  $x = 0$  and  $x = 1$ . Setting  $x$  to these values gives

$$c_1 + c_2(0) = 0, \quad c_1 + c_2(1) = 0.\tag{16}$$

which gives  $c_1 = c_2 = 0$ .

Show that  $\{1, x, 2x - 3\}$  are not linearly independent. We consider

$$c_1(1) + c_2x + c_3(2x - 3) = 0. \quad (17)$$

Can we choose  $c_1, c_2$  and  $c_3$  such that (17) is true without having all the constants zero.

Regrouping (17) gives

$$(c_2 + 2c_3)x + c_1 - 3c_3 = 0, \quad (18)$$

and as long as we choose

$$c_2 + 2c_3 = 0, \quad c_1 - 3c_3 = 0, \quad (19)$$

then (17) is identically zero and there are many nonzero choices for  $c_1, c_2$  and  $c_3$  so the answer is no,  $\{1, x, 2x - 3\}$  are not linearly independent.

### Wronkian

There is a very convenient way to show linear independence. We define the Wronskian as

$$W(f(x), g(x)) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - f'(x)g(x). \quad (20)$$

Here, we consider the Wronskian of the previous example where  $f(x) = 1$  and  $g(x) = x$  so

$$W(1, x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1. \quad (21)$$

For those that have taken Linear Algebra, this is the usual determinant.

Two functions are said to be linearly independent if

$$W(f(x), g(x)) \neq 0 \quad \text{for all } x. \quad (22)$$

Are the function  $\{\sin x, \cos x\}$  linearly independent.

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0 \quad (23)$$

so yes, linearly independent.

Are  $\{x^2, -x^2\}$  linearly independent? Since

$$W(x^2, -x^2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = x^2(-2x) - 2x(-x^2) = 0 \quad \text{for all } x \quad (24)$$

so linearly dependent.

Are  $\{x^2 + x, x^2 - x, x\}$  linearly independent? Now we have three functions so we need to go to  $3 \times 3$ . Since

$$\begin{aligned}
 W(x^2 + x, x^2 - x, x) &= \begin{vmatrix} x^2 + x & x^2 - x & x \\ 2x + 1 & 2x - 1 & 1 \\ 2 & 2 & 0 \end{vmatrix} \\
 &= (x^2 + x) \begin{vmatrix} 2x - 1 & 1 \\ 2 & 0 \end{vmatrix} - (x^2 - x) \begin{vmatrix} 2x + 1 & 1 \\ 2 & 0 \end{vmatrix} + x \begin{vmatrix} 2x + 1 & 2x - 1 \\ 2 & 2 \end{vmatrix} \\
 &= (x^2 + x)(0 - 2) - (x^2 - x)(0 - 2) + x(2(2x + 1) - 2(2x - 1)) \\
 &= -2(x^2 + x) + 2(x^2 - x) + 4x = 0
 \end{aligned} \tag{25}$$

so linearly dependent.

So now we focus on solving the following constant coefficient ODE

$$ay'' + by' + cy = 0, \quad a, b, c \text{ are all constant} \tag{26}$$

and then later solving

$$ay'' + by' + cy = f(x), \quad a, b, c \text{ are all constant, } f(x) \text{ prescribed.} \tag{27}$$