

# Spring 2026 – Math 3331

## Reducible Second Order ODEs

In general, second order ODEs are of the form

$$y'' = F(x, y, y'). \quad (1)$$

For example,

$$y'' + y' = 0, \quad y'' + \frac{y'}{x} = 0, \quad y'' + \sin(y') = x, \quad (2)$$

and

$$y'' + y = 0, \quad y'' + y' + y^2 = 0, \quad y'' + \sin(y) = 0 \quad (3)$$

are all second order ODEs. But there is something special about the two sets. In the first set, they all have  $y$  missing whereas in the second set, they all have  $x$  missing. These special ODEs can actually be reduced to first order ODEs. The following examples illustrate.

### Missing $y$

In this case we have

$$y'' = F(x, y') \quad (4)$$

If we let  $y' = u$  then  $y'' = u'$  and (4) becomes

$$u' = F(x, u) \quad (5)$$

a first order ODE and the methods introduced for solving first order ODEs can be used. The following examples illustrate.

*Example 1.*

Solve

$$y'' - y' = 1 \quad (6)$$

If  $y' = u$  then  $y'' = u'$  then (6) becomes

$$u' - u = 1 \quad (7)$$

a separable ODE. Separating and integrating gives

$$\frac{du}{u+1} = dx \quad \rightarrow \quad \ln|u+1| = x + \ln c_1 \quad (8)$$

or

$$u = -1 + c_1 e^x. \quad (9)$$

Now, we replace  $u = y'$  so

$$\frac{dy}{dx} = -1 + c_1 e^x. \quad (10)$$

and integrate once more

$$y = -x + c_1 e^x + c_2. \quad (11)$$

Notice that there are two constants of integration  $c_1$  and  $c_2$ . This is typical in solving second order ODEs.

*Example 2.*

Solve

$$y'' + \frac{y'}{x} = 2, \quad y(1) = 1, \quad y'(1) = 2. \quad (12)$$

If  $y' = u$  then  $y'' = u'$  then (12) becomes

$$u' + \frac{u}{x} = 2 \quad (13)$$

a linear ODE. The integrating factor is

$$\mu = \exp\left(\int \frac{1}{x} dx\right) = \exp(\ln(x)) = x \quad (14)$$

so

$$x \left(u' + \frac{u}{x}\right) = 2x \quad \rightarrow \quad \frac{d}{dx}(xu) = 2x \quad (15)$$

and separating and integrating gives

$$xu = x^2 + c_1 \quad (16)$$

At this point, let's bring in the IC  $y'(1) = 2$  so  $u(1) = 2$  this gives  $(1) \cdot (2) = 1^2 + c_1$  so  $c_1 = 1$  giving

$$xu = x^2 + 1 \quad \text{or} \quad u = x + \frac{1}{x} \quad (17)$$

Now back replace  $u = y'$  giving

$$y' = x + \frac{1}{x} \quad (18)$$

Now integrate again giving

$$y = \frac{1}{2}x^2 + \ln(x) + c_2. \quad (19)$$

The second IC  $y(1) = 1$  gives  $1 = 1/2 + c_2$  gives  $c_2 = 1/2$  and we obtain

$$y = \frac{1}{2}x^2 + \ln(x) + \frac{1}{2} \quad (20)$$

### Missing $x$

In this case we have

$$y'' = F(y, y') \quad (21)$$

If we try using  $y' = u$ , then  $y'' = u'$  and we obtain

$$u' = F(y, u). \quad (22)$$

However,  $u' = \frac{du}{dx}$  so we have

$$\frac{du}{dx} = F(y, u) \quad (23)$$

and we have  $x, y$  and  $u$  in the problem. But we can use the chain rule from Calc 1

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u \quad (24)$$

and (23) becomes

$$u \frac{du}{dy} = F(y, u), \quad (25)$$

a first order ODE. The following examples illustrate.

*Example 3.*

Solve

$$y'' + \frac{y'^2}{y} = 0. \quad (26)$$

If  $y' = u$  then  $y'' = u \frac{du}{dy}$  then (26) becomes

$$u \frac{du}{dy} + \frac{u^2}{y} = 0 \quad (27)$$

If  $u \neq 0$  we obtain a separable ODE

$$\frac{du}{u} = -\frac{dy}{y} \quad (28)$$

and integrating gives

$$\ln |u| = -\ln |y| + \ln c_1 \quad (29)$$

and exponentiating

$$u = \frac{c_1}{y} \quad (30)$$

Now  $u = y'$  so

$$\frac{dy}{dx} = \frac{c_1}{y} \quad (31)$$

or

$$y dy = c_1 dx \quad (32)$$

or

$$\frac{1}{2}y^2 = c_1x + c_2 \quad (33)$$

As  $c_1$  and  $c_2$  are arbitrary constants, we can absorb the 2 giving

$$y^2 = c_1x + c_2 \quad (34)$$

Suppose we we given IC/BC say  $y(0) = 0$  and  $y(1) = 1$  then we use these in (34) so

$$0^2 = c_1(0) + c_2, \quad 1^2 = (1)c_1 + c_2 \quad (35)$$

giving  $c_1 = 1$  and  $c_2 = 0$  leaving the final solution as

$$y^2 = x \quad (36)$$

Checking, if  $y = x^{1/2}$  then  $y' = \frac{1}{2}x^{-1/2}$  and  $y'' = -\frac{1}{4}x^{-3/2}$  and substituting into our original ODE gives

$$y'' + \frac{y'^2}{y} = -\frac{1}{4}x^{-3/2} + \frac{\left(\frac{1}{2}x^{-1/2}\right)^2}{x^{1/2}} = -\frac{1}{4}x^{-3/2} + \frac{1}{4}x^{-3/2} = 0 \quad \checkmark \quad (37)$$

*Example 4. (long example)*

Solve

$$y'' - y = 0. \quad (38)$$

If  $y' = u$  then  $y'' = u \frac{du}{dy}$  then (38) becomes

$$u \frac{du}{dy} - y = 0 \quad (39)$$

Separating gives

$$udu = ydy \quad (40)$$

and integrating gives (the 1/2 is absorbed into  $c_1$ )

$$u^2 = y^2 + c_1 \quad (41)$$

Now  $u = y'$  so

$$\left(\frac{dy}{dx}\right)^2 = y^2 + c_1 \quad (42)$$

and separating

$$\frac{dy}{\sqrt{y^2 + c_1}} = dx. \quad (43)$$

Integrating

$$\ln |y + \sqrt{y^2 + c_1}| = x + \ln c_2 \quad (44)$$

Now for some fun algebra

$$\begin{aligned} y + \sqrt{y^2 + c_1} &= c_2 e^x \\ \sqrt{y^2 + c_1} &= c_2 e^x - y \\ y^2 + c_1 &= (c_2 e^x - y)^2 \\ y^2 + c_1 &= c_2^2 e^{2x} - 2c_2 e^x y + y^2 \\ c_1 &= c_2^2 e^{2x} - 2c_2 e^x y \\ 2c_2 e^x y &= c_2^2 e^{2x} - c_1 \end{aligned} \quad (45)$$

and solving for  $y$  gives

$$y = \frac{c_2}{2} e^x - \frac{c_1}{2c_2} e^{-x} \quad (46)$$

or

$$y = c_1 e^x + c_2 e^{-x} \text{ (new constants } c_1 \text{ and } c_2) \quad (47)$$

As we will find next week, solving ODE such as

$$ay'' + by' + cy = 0, \quad a, b, c \text{ constants} \quad (48)$$

will be much easier to solve.