Polynomial Least Squares and Sovereign Debt Risk Indicators

Manuel Tarrazo

Abstract

In this study we apply polynomial least squares (PLS) to the computation and analysis of quickly changing economic and financial data undergoing intense change processes. PLS are well-known in engineering by their ability to obtain maximum information from few data points. PLS models are nonlinear but they can be estimated with ordinary least squares because the variables remain linear in the parameters. The last twenty years have seen an explosion of public and private debt, which has come to float in recessionary economies. Enhancing fixed income risk analysis has therefore become critical. Our initial application to the 10-year government notes series of Greece, Italy, Spain, and Portugal indicates this to be the case. The methods can be applied to any variable to detect resting points, attractors, and other dynamic properties, which makes PLS methods well-suited to the analysis of economic and financial data from emerging countries.

Key words: Polynomial least squares, parabolic least squares, polynomial interpolation, duration, sovereign debt, European debt crisis, price-yield-to-maturity.

This study makes three contributions. First, it shows a straightforward way to compute bond indicators (duration, convexity) using polynomial least squares (PLS). The relationship between the yield-to-maturity and the price is non-linear, and PLS are one of the few linear tools that can be applied to nonlinear relationships. From a pedagogical viewpoint, PLS methods help students develop familiarity with EXCEL and become aware of the interplay between estimation and optimization. There are so many ways in which duration can be computed that yet another method would not be noteworthy unless it offered something other methods do not, which is the case with respect to our other two contributions.

Second, PLS methods offer clear advantages modeling fixed income securities. PLS are most economical in data points needed (three, quadratic; four, cubic), and one can study higher order moments such as asymmetry and kurtosis, as well. But the most interesting feature is the ability to focus solely on the fewest, most recent data points. In a way, PLS methods magnify the information captured by recent data points, as if looked at through a magnifying glass. This feature may allow analysts to spot trend changes –a characteristic presumably most helpful when dealing with rapidly changing data.

The third contribution of this paper is the application of PLS methods to European sovereign debt data, i.e., 10-year bond yields for a few of the countries that have experienced the highest fluctuations in their public debt –Greece, Italy, Spain, and Portugal. This context reveals another interesting feature of PLS: signals can be extracted from empirical observations quickly, without transforming the data, and without presupposing any theoretical relationship

We first apply empirical methods to the theoretical relationship and, after determining which specification captures that relationship the best. We then apply those equations to empirical data. The search for the best polynomial specifications is contained in the first part of the study, and the second focuses on the application to actual fixed income data --the 10-year government note from Greece, Portugal, Italy, and Spain. These are time series that register a level of tension and stress as few others since the aftermath of the 2008 crisis. The data was kindly provided by the International Monetary Fund (IMF).

This study does not represent the first time that polynomial processes have been applied to bond prices. Rodriguez (1998) discussed simple linear and quadratic functions within the context of finding an approximation to the yield to maturity. Crack and Nawalkha (2000) used a polynomial approximation for the yield curve, which they used to study the sensitivity of bond risk measures to changes in the level, slope, and curvature of hypothetical yield function. The difference in the approach itself is, for us, more significant than showcasing a particular function or technique. And the approach is to focus on empirical analysis. For example, Willner (1996) uses terms and concepts such as "key rates," "bogeys," "level, slope, curvature," and "hump" in the way we have used them. These terms and concepts, according to Willner, "introduce another approach to meet the portfolio manager's need for intuitive, descriptive, and comprehensive risk exposure information," (op. cit., p. 49). It is also interesting to note that meeting practical needs may not require statistical sampling or complicated stochastic measures, but rather only modest calculus tools.

Fixed income debt is one of the "legs" of modern financial systems. Public debt has registered an unprecedented peacetime growth in public debt, which has become one major tool in economic policies –see for example The International Monetary Fund's (2015) recent seminar entitled "The Political Economy of High Debt." Add to the already existing programs the European Central Bank launching of a 1 trillion euro rescue plan. The size of public debt makes it impossible for it to be managed only at the public institutions level (domestic and international). One would think that private markets would need to be involved, more or less visibly. This means analysts must closely follow the decision of government and international organizations and, especially, they must scrutinize markets reactions. In this context, our findings are most encouraging and support the usefulness of PLS methods, which could end up being routinely used not only in the classroom, but also to extract expectation dynamics from most economic and financial data.

A Polynomial least squares and fixed-income risk indicators

Our research strategy is summarized in Exhibit I. We can distinguish two frameworks when learning about investing and managing investment portfolios: the theoretical and the empirical.

Refer Exhibit I

The theoretical framework is the first one we encounter in textbooks and focuses on abstract /ideal concepts and idealized relationships amenable to calculus-based, exact, formulae. When we try to learn by doing, as when managing an actual investing portfolio, we work in a different framework --the empirical one-- where we must wrestle with actual conditions, imperfect relationships and with the limitations of extant knowledge.

Applying approximate methods when we carry the analysis in the theoretical/abstract/ideal realm seems accessory and wasteful. However, approximate methods are the only ones that enhance our knowledge of whether and how theoretical relationships work in practice. Given our limited knowledge, practical effort can rarely make use of the perfectly defined concepts that flawlessly embody structural relationships in models that appear to be sufficient for decision-making. In this study we transition from the theoretical into the empirical realm. First, we determine what the approximate model specifications are that best capture the essence of the theoretical model and, afterwards, we deploy them in the empirical realm.

In this first section of the study we take care of the first two steps of such a transition: a review of the theoretical knowledge and the identification of the approximate specification that best captures theoretical properties. We will observe that a cubic equation may best represent the theoretical relationship between the price of a bond and its yield-to-maturity, which can be studied using polynomial least squares of third order. When we try to fit such an approximate equation to ideal data, we discover that, in general, the more the observations the better the fit. The problem is that in the empirical realm successive observations may bring information of different states and structures, and simply accumulating observations may actually mislead us and bias our decision-making. One of the major reasons to deploy PLS is that they work with a minimum of observations. The second part of the study will focus on the application.

A.1 Fixed income and risk indicators

Fixed income investments provide contractual payments to their owners, usually in the form of coupon payments and price appreciation. Contrary to equity investing, as time passes and the maturity draws nearer, uncertainty concerning market values and resale prices diminishes. Coupon payments represent, in general and especially for long maturity bonds, the most important cash flow to bondholders. Moreover, these intermediate coupon revenues can be reinvested. Investors in fixed income securities care about 1) receiving the specific, contractual stream of cash-flows, 2) the market value of their holdings, and 3) the expected returns over the life of the investment, which means that they care both about cash flows as well as about changes in interest rates. An increase in interest rates lowers the market value of existing fixed income contracts, also raising reinvesting rates. Investors care about interest rate volatility as well because the very same set of movements in a given interest rate will affect bond holdings with different characteristics (rating, maturity, coupon, and so on) in different ways.

Risk analysis starts by studying the relationship between a bond’s price and its yield to maturity, which is summarized in the yield to maturity equation:

$$p = \sum_{t=1}^{t=T} \frac{cr * face}{(1 + ytm)^t} + \frac{face}{(1 + ytm)^T} \quad (1)$$

where

T = maturity of the bond
cr = coupon rate
face = face value of the bond
p = market price of the bond

Duration and convexity are indicators of risk in fixed income investing. Duration is an indicator of the effects of a change in the reinvestment rate on coupon cash flows and the market value of the investment. Convexity is an indicator of the effects of interest rate volatility on bond values.

Duration and convexity are obtained from Equation (1) by calculating its derivatives analytically or numerically. Let x, and p = f(x), represent the yield to maturity and price of the bond, respectively. It can be shown —see Fabozzi (2000, Chapter 4), or De La Grandville (2001, Chapter 4) — that

$$\text{duration} = d = f'(x)/p \quad (2)$$

$$\text{dollar duration} = f'(x) = d * p \quad (3)$$

$$\text{convexity} = cvex = f''(x)/p \quad (4)$$

$$\text{dollar convexity} = f'(x) = cvex * p \quad (5)$$

where d represents what is known as modified duration throughout this note.

Technically, duration represents the number of periods the bondholder has to wait to earn his/her expected return. For pure discount bonds (aero-coupon bonds) the duration is, therefore, equal to the maturity of the bond. Holding the bond for its duration theoretically guarantees earning the expected yield-to-maturity, if interest rates are such that the yield to maturity curve only experiences parallel changes. The reinvestment of the coupons has a very large effect on actual investment returns (realized compound yields).

As noted above, both duration and convexity are derived analytically by obtaining the first and second derivatives of the relationship between a bond's price and its yield to maturity. Numerically, durations and convexities can be calculated in at least the following three different manners:

1. A simple table computation. This approach leads to a tabular computational format where each periodic payment due to the bondholder is weighted in a particular way, see Fabozzi (2000). Popular spreadsheet software helps in computing such measurements; one can start calculating the indicators for the longest maturity bond, after which calculating indicators for other bonds becomes a matter of "copy and paste."
2. A closed-form, exact formula, which provides one-step computation. In the case of duration, a very handy closed-form is due to Hawawini (1982, Chapter 2), and another one to Chua (1985, 1984). Closed-form formulas are as old as the duration concept itself —Maucaulay provided a closed-form for duration, see Smith (1988) and references therein. For convexity, one can use Brooks and Livingston's equation (1989), or Blake and Orszag's (1996). Research in finding closed-form approximations offers benefits other than simplifying numerical computations. Babcock (1985) shows that duration can be interpreted as a weighted average of two factors, one related to reinvesting and the other to capital appreciation. Closed-form formulae also help to study the relative

importance of duration and convexity in ascertaining bond price changes, see Brooks and Livingston (1992).

3. A graphical-based method. A very simple way to compute duration and convexity is what we shall refer to as the price-spread method, which uses the curvature of the price-yield to maturity relationship and can be easily explained in the classroom with the help of graphs, see Heck, Zivney, and Modani (1995).

A default-free and bankruptcy-free risk bond is still a risky investment that must be evaluated appropriately. The investor knows with certainty the price to be paid in order to acquire the bond, the coupon payment to be received each period, and the nominal (face) value to be received at the maturity of the bond. The indicators of returns (current yield, and yield to maturity) depend on changes of reinvesting opportunities (if the coupons are to be reinvested), and changes in the price of the bond (if bonds could be sold). Duration and convexity help to assess some of these potential changes. For example, nearly every instructional textbook on fixed income investing shows how to use duration and convexity formulate in conjunction with Taylor's expansion to approximate price changes due to changes in the yield. Equations (6) and (7) show how to calculate the approximate dollar change and the relative change, respectively

$$dp = d * p * dx + 1/2 * cvex * p * dx^2 \tag{6}$$

$$dp/p = d * dx + 1/2 * cvex * dx^2 \tag{7}$$

where dp and dx represent the incremental change in price and yield, respectively ("d" stands for delta, Δ).

Note, first, that the yield to maturity assumes that all reinvestments of coupons will be made at the very same rate --at the yield to maturity itself. This means we sometimes use a single equation sometimes to assume a fixed price and to allow a variable yield to maturity, and at other times we assume a given, unchanging yield to maturity to price the bond. Second, we may not be that interested in the relationship between the price and the yield to maturity in itself. According to equation (1), the yield to maturity is in fact a dependent variable and should not be used to predict prices. However, we know that clusters of yields move together. Therefore, we ultimately want to anticipate changes in one or more market yields that in practice do affect the market price of the bond. In other words, the yield to maturity is used as an indicator of the evolution of that cluster of "interest rates" affecting our fixed income holdings. Third, duration, convexity, and the expected price change are all approximations as well. Textbooks refer to the relationship between the bond's price and yield to maturity as "actual," but this relationship is also an approximation that, like duration and convexity, assumes unchanging reinvestment rates (a flat yield curve allowed to shift up and down), and very small (infinitesimal, in fact) yield changes. Equation (1) is an abstract idealization. The market relationship between the price of a bond and interest rates is likely to be very complex, so complex that, in fact, we do not have a functional form for it yet and a partial difference equation would not fully explain everything we would like to know. In this situation, fixed income investors welcome indicators of risk, even if they provide no more than limited clues to the behavior of prices and interest rates in the neighborhood of current values. That is why polynomial least squares may be found to have some usefulness.

A.2 Polynomial least squares, duration and convexity

As the name suggests, polynomial regression indicates fitting a polynomial of a given order to a series of observations. A polynomial is a group of terms where the variables interact in an additive or multiplicative manner and are themselves raised to integer powers. One comes across them in regression and in interpolation where polynomials are used to find the perfect fit (R-squared = 1) with the minimum number of observations.

Polynomials appear naturally after some operations. For example, a second order Taylor approximation reminds us of a quadratic curve, which is itself a polynomial of second order. It turns out this second order polynomial provide a very easy way to calculate duration and convexity because the quadratic function is well--known and very easy to manipulate. Let us see, using a second order of Taylor expansion of equation (1) is equivalent to using this function:

$$p = f(x) = a + b\,x + c\,x^2 \tag{8}$$

Its derivatives, and therefore duration and convexity, are easily calculated:

$$f'(x) = b + 2\,c\,x \tag{9}$$

$$f''(x) = 2\,c \tag{10}$$

$$d = f'(x) / \text{price} = (b + 2\,c\,x) / \text{price} \tag{11}$$

$$\text{cvex} = f''(x) / \text{price} = 2\,c / \text{price} \tag{12}$$

The quadratic approximation is used by Heck, Zivney, and Modany (1995) to compute durations and convexities in a classroom friendly manner, which is easily explainable with the help of a graphic of the price-ytm theoretical relationship. We can use their example to illustrate the application of quadratic, polynomial least squares (PLS) specification. We first pick up a bond: T (maturity) = 18, CR (coupon rate) = 12% annual (paid semiannually), and selling at p = 1264,990, which provides a yield of 9% annually. Now we compute the price for two other yields to maturity, for which we can use a simple 10 basis point spread (0.089, 0.091); we obtain two other prices (1275.6592, 1254.4582). Now we have three yields and three prices, and we would form the PLS equations. That is, rearrange this data

obs	ytm	p
1	0.089	1275.659241
2	0.090	1264.990609
3	0.091	1254.458218

to form the regression matrices:

$$\underbrace{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}}_{X} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \underbrace{\begin{vmatrix} p_1 \\ p_2 \\ p_3 \end{vmatrix}}_{Y} \tag{13}$$

as

obs	intercept	x = ytm	x^2 = ytm^2	price
		X		Y
1	1	0.089	0.007921	1275.6592
2	1	0.09	0.0081	1264.9906
3	1	0.091	0.008281	1254.4582

Then we can calculate a, b, and c with the usual OLS matrix inversion:

$$\begin{vmatrix} a \\ b \\ c \end{vmatrix} = (X'X)^{-1} X'Y = \begin{vmatrix} 2770.817627 \\ -22862.30957 \\ 68121.10156 \end{vmatrix} \qquad (14)$$

These are the values for the derivatives, duration, convexity, estimated prices (p-hat), errors, squared errors, and mean-squared-error (MSE), respectively:

p-hat	f'(x)	duration	error	error^2
1275.659321	-10736.75349	8.416631295	-7.94179E-05	6.30721E-09
1264.990688	-10600.51129	8.379913034	-7.96239E-05	6.33997E-09
1254.458298	-10464.26909	8.341664102	-7.98924E-05	6.38279E-09
			MSE	9.51498E-09
	f''(x)	cvex		
	136242.2031	107.7021459		

Heck, Zivney, and Modani (1995) obtain the following values for the 9% annual yield to maturity after some rounding: modified duration = 8.38 years, convexity = 107.70. Even by running this simple example, one notices that precision is very important. An innocent rounding to two decimals (e.g., 1265 instead of 1264.990) will not provide the correct numbers.

A cubic (polynomial of order 3) function can be expressed as:

$$p = f(x) = a + b\,x + c\,x^2 + d\,x^3 \qquad (15)$$

The cubic approximation needs at least four data points. The fourth point may bring information about asymmetry. It is very easy to specify higher order terms for the price-rate relationship by simply adding additional powers of the rate to the right-hand-side of the basic data matrix (this is called "nesting" in econometrics). In the four-observation cubic model, the data matrices would comprise one equation per observation and four columns for X, each one containing an additional power for the independent variable, in addition to the intercept:

$$\underbrace{\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix}}_{X} \begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = \underbrace{\begin{vmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{vmatrix}}_{Y} \qquad (16)$$

Solving the OLS "normal" equations would provide the solution values for the estimates:

$$[\,a\;\; b\;\; c\;\; d\,]' \quad = \quad (X'X)^{-1}\, X'Y \tag{17}$$

Duration and convexity would be approximated, as usual: i.e., by calculating first and second derivatives. As noted earlier, convexity would not be a constant but, instead, would change throughout the curve.

$$d = f'(x) \,/\, price = (b + 2\,c\,x + 3\,d\,x^2) \,/\, price \tag{18}$$

$$cvex = f''(x) \,/\, price = (2\,c + 6\,d\,x) \,/\, price \tag{19}$$

These would be the equations for a quartic specification:

$$p = f(x) = a + b\,x + \; c\,x^2 + d\,x^3 + e\,x^4 \tag{20}$$

$$d = f'(x) \,/\, price = (b + 2\,c\,x + 3\,d\,x^2 + 4\,e\,x^3) \,/\, price \tag{21}$$

$$cvex = f''(x) \,/\, price = (2\,c\,x + 6\,d\,x^2 + 12\,e\,x^3\,) \,/\, price \tag{22}$$

The computations to solve PLS equations can be performed with a variety of methods: econometric and statistical software packages, mathematical packages (e.g., MATLAB), or programmed in a variety of languages. Spreadsheet packages like EXCEL offer several methods as well: matrix operations, dta analysis tools, functions to directly obtain the coefficients (through a combination of the INDEX and LINEST functions), and even a way to plot the yield-price relationship as an x-y chart, and by clicking on the curve, one can add a trend and attach the equation and R-squared to the chart.

Up to this point we have presented the theoretical price-yield to maturity relationships and we have noted some of its limitations. We have also reviewed PLS methods and indicated that they may help to better ascertain in practice the empirical relationship between price and yields from which risk indicators such as duration and convexity are extracted. Studying PLS methods in the context of the theoretical price-yield relationship offers these findings:

a) Exhibit II shows that the theoretical relationship stretches over the whole range of the price-yield relationship, but the quadratic approximation fails to provide an unbiased fit: it tends to underestimate the effects of large changes and to overestimate the effects of small price-yield changes. Even the theoretical relationship does not have a constant slope and seems to call for a higher order approximation. PLS offers a model that could help in this case because it is nonlinear and can use higher order terms. But it still retains linearity in the parameters, which simplifies its application.

b) Exhibit III studies the price-yield relationship in an empirical context. Let us suppose, for example, that the investor has recorded one observation at a time and has tried to ascertain the duration and convexity as the information has become available. This simple exercise shows that the cubic approximation may be able to capture linear and nonlinear effects. The estimation is quite sensitive to new observations. The exercise also highlights the ability of the model to both use the minimal number of observations and to make maximum use of the information provided by each of them.

c) Exhibit IV presents some visualizations of the way to understand the application of PLS in an empirical context. Figure A shows that, in a quadratic setting, because a curve can touch any two points, it is very easy to link curves to observations in an arbitrary manner. In an empirical setting, it is best to visualize the price-yield relationship as a series of patches which, in turn, offers to us two important options: 1) the ability to adequately represent changes at different tranches in the price-yield in situations when we are not sure whether the structures have changed; and b) the possibility to price (assess, evaluate) key empirical yields beyond the theoretical yield-to-maturity rates.

In the next section we apply the empirical framework to the actual time data of 10-year sovereign notes for some European countries recorded during an economic climate of intense stress and policy changes.

Refer Exhibit II, III, IV

B. Application: Sovereign Debt Crisis

Our application is organized into three subsections focused on providing a context for the analysis, studying the results from applying PLS methodology to the selected data, and offering some suggestions and guidance for further research, respectively.

B.1 Sovereign debt in the context of the European crisis

The National Bureau of Economic Research (NBER) and the Centre for Economic Policy Research (CEPR) study the evolution of economic variables and provide dates signaling the highs and lows of economic activity, otherwise known as “peaks” and “troughs” of the business cycle. Both the NBER and the CRPR date the first quarter of 2008 as the highest point of economic expansion, also signaling the onset of the most recent wide-scale recession.

The severity of the economic crisis has been highlighted by the intensity of each of its facets -- real, monetary, and in the international linkages (trade and capital flows).The clear evidence represented by the turning point in the business cycle and in other real variables, especially employment, was anticipated by violent outbursts in the financial realm. Some of the earliest trouble was detected in real estate markets and their wildly overgrown associated securities (collateralized debt obligations, mortgage-backed securities, and so forth). In June 2007 in the U.S., Bear Stearns let two funds fail --their value, assessed at $18 billion in December 2006, had disappeared barely six months later. Apprehension gripped economies all over the world, as it was feared they could collapse like a house of cards built on mountains of shaky securities. In March 2008, Bear Stearns was force-sold to J. P. Morgan. In August, BNP Paribas froze (that is “halted withdrawals”) from three funds because the lack of trading had made them impossible to price. In September, two federal entities, the Federal National Mortgage Association and the Federal Home Loan Mortgage Association, were seized by the US government. Barely a week later, Lehman Brothers was “advised” to file for bankruptcy proceedings. Excellent accounts of the financial crisis and its securitization aspects are Chincarini (2012) and Kolb (2011), both of which include annotated timelines of the crisis.

Distress in "real" economic variables added to the distress in the financial markets, and political turmoil also made its presence known with violent demonstrations and u-turns in political elections. The management of the crisis required full attention and coordination between international organizations (usually the International Monetary Fund, The European Association, and the European Central Bank) and specific countries, as matters took unexpected and worrisome turns in a string of countries: Greece, Italy, Spain, Ireland, and Portugal, Iceland, and so on.

The fall in economic activity brought about devastating increases in unemployment, which crippled and hurt certain European economies (e.g., Spain, Greece) at levels unseen by some of its younger generations, and which brought to older generations memories of the extremely difficult times suffered during the oil crises of the 1970's. Making things worse, a familiar mechanism to weather economic storms became unavailable for ailing economies: some governments had enormous public expenses that kept increasing, especially when tax revenues were collapsing. At least in retrospect, it seems that instead of making the best of the good times by adding a measure of prudence, private and public decision makers used debt and expenses to push their household and national economies until they literally burst at their financial seams. This meant that yet another serious dimension of the crisis made its entrance in world affairs: sovereign debt troubles. A debt crisis usually refers to situations where the borrower cannot pay, either for a while (defaults, bankruptcies), or presumably forever (dissolution). Up to some point during the crisis, the fear focused on the payment ability of private borrowers such as households in mortgage loans, and liquidity problems in the case of private businesses (commercial and investment banking). In 2010, however, whole countries appeared to be insolvent.

Two ensuing prospects had to be tackled. One was to create a support mechanism to help the affected countries help themselves. The second was to reinforce the wider, multicounty mechanism which, in the case of the European Union, was originally created to enhance each country's prospects. In 1991, the Maastricht Treaty created a monetary union culminating in the 1999 adoption of the euro as a unique currency by eleven founding members. The economic European crisis became a crisis of the euro as well.

In sum, the still raging European crisis --and more in the case of Greece than anywhere else-- has required a great deal of effort in three major areas --fiscal integration, monetary convergence, and international trade and capital flows. The 10-year government note is one indicator at the very middle of the economic storm, as it reflects political and economic decisions, fiscal situations, and financial issues concerning the monetization of public debt and international private and public capital flows.

In the rest of this section we will apply polynomial least squares methods to the figures of the 10-year government note for selected European countries.

B.2 Application to specific countries

The only data needed to calculate durations and convexities with polynomial least squares are the prices (or par values) and coupon rates. From this information, one can calculate yields-to-maturity, since we know the notes have a 10-year maturity. For specific cases, the data is readily available in some sources (e.g., The Wall Street Journal), but it would be prohibitively laborious to obtain by hand the 790 daily data points covering the 5/1/10-7/1/12 period of interest, for several countries. The data is available from the following time series in the International Financial Statistics (IFS) database, which were most kindly made available to the author by the International Monetary Fund (Petra Dacheva, Martine Guerguil, and Gerd Schwartz):

- YCGT0156 Index. Greece Yield Curve, 10Y
- YCGT0040 Index. Italy Yield Curve, 10Y
- YCGT0061 Index. Spanish Yield Curve, 10Y
- YCGT0084 Index. Portugal Yield Curve, 10Y

Each series included the date, the underlying security identifier, the yield-to-maturity, the price, and the coupon. Using polynomial least squares (PLS), we computed empirical duration and convexity for both the quadratic and the cubic specifications for each of the four countries above. The data for Ireland was incomplete for the aforementioned period and the results have not been included in this study.

Note that the quadratic PLS specification uses three historical data points, while the cubic specification uses four historical data points. It is easy to set up a program in MATLAB or in EXCEL to calculate the rolling values for durations and convexities. The data visualization tools provided by EXCEL, in addition to its spreadsheet design make it especially suitable to scrutinize the evolution of the data over time and at specific time periods, such as those related to debt negotiations.

It is appropriate in this study to focus on those findings that highlight the capabilities added by the methods we propose. Policy analysis is only used to provide brief contextual reminders of certain historical periods of interest.

The following are general findings obtainable with any of the time series we employed:

1. As expected the calculations of empirical duration and convexity are very sensitive to new data points. Further, they reinforce the patch-like structure of the data as it comes day-by-day (Figure B in Exhibit IV). In other words, it may be grossly inaccurate to assume that the new data points belong to an idealized existing curve.
2. The cubic specification certainly adds something to the computation of duration. In all cases, it corresponds to the more jagged curve.
3. To the contrary, the quartic specification does not seem to add much to the information provided by the cubic equation. Many of the coefficients for the fourth order effects are nil.
4. The convexity roughly corresponds to the square of the duration and, as such, it simply magnifies the changes already observed in the duration series without adding any additional information.
5. In situations of specific structural change, PLS computations for durations --both quadratic and cubic-- show extreme values, for example, a value of 40 years instead of the theoretical value of seven, or even negative values, which then have to be filtered out of the data. These values are not wrong and actually imply a break in the underlying economic logic of the data. This is most interesting to observe when analyzing the effects

of changes in the coupon rates, which are not used directly in the computation of durations and convexities. However, after calculating durations and convexities using PLS we suddenly observe a set of four or six nonsensical values in a row that come back to plausible ones, which can be traced back to a change in the coupon rate. An example of such an event is given below. The Italian 10-year note changed its coupon rate from 4% to 3.75% and the end of August, 2010. Immediately, the values for duration and convexity computed with PLS methods register changes indicating a rupture on the data series. In other words, even if we did not have data on coupon rates, we could detect that something has changed given the behavior of duration and convexity estimates. This happening strengthens the case for using PLS estimates as a monitoring device.

This is a numerical example showing the effect of changes in coupon rates on duration estimates (Italian 10-year government note):

Date	Coupon	yield	price	d-quad	d-cubic
08/29/10	4	0.0376	102.276	8.305323224	8.310845
08/30/10	4	0.03779	102.115	8.285114993	8.298178
08/31/10	3.75	0.03812	99.775	**-14.0201541**	**-14.3716**
09/01/10	3.75	0.03785	100.003	**405.8802786**	**322.4922**
09/02/10	3.75	0.03787	99.982	6.238481027	**-215.877**
09/03/10	3.75	0.03793	99.932	11.04133543	8.961809

One more item of interest before proceeding to review country results: in 2001, two years after the adoption of the Euro, these were the approximate yields-to-maturity for the 10-year government note for some of the countries we will review: Greece, 5.36; Spain, 5.09; Italy, 5.16; and Germany, 4.85%. The idea of fostering and maintaining a “convergence” in financial indicators that, in turn, is necessary for monetary and exchange rate stability seemed to have worked during the 2000-2007 period, but afterwards it broke down. A 5% yield-to-maturity target on a 10-year note implies a coupon rate of 5%, if sold at par value, and duration of 5.17 years (annual computations). As noted earlier, the duration can be thought of as the waiting period, and the “patience” the investor must have, for achieving the expected return when buying such a note --it includes cash flows from coupons, reinvestment of the coupons, and selling the bond the bond after exactly 5.7 years from the time of the purchase transaction settlement date.

Focusing on the durations has some advantages over simply looking at daily par value-yield quotes. An increase in the coupon rate, coupled with a drop in the price of the bond, can keep a yield in range without being very noticeable. The duration indicators show at one glance the effects of changes in one or more terms in the note and, therefore, are a more reliable indicator of the current patience of investors with respect to such financial assets.

Exhibit V shows the calculation for Greece (the ninth largest economy of the Eurozone with a 2.04% of the 9.487 trillion euros GDP of the 17 countries-Euro area; www.eurostat.com , GDP and main components - Current prices [nama_gdp_c], year 2012 figures). The sovereign Greek crisis became “official” when the recently newly elected government in October 2009 declared that the previously released economic data was not reliable. After feverish international work, it triggered a first emergency summit on Greece in February 2011. Revisions of debt figures and a painful adjustment process ensued. In sum, the 10-year note numbers tell a story of inexorable debt deterioration that has miraculously avoided full-fledged, outright default. The numbers started in May 2010 with a 6.25% coupon, selling at about 80% par values, and with yields between 8-9%, offering durations around seven years. The series ended selling at 15-20% of nominal value, 2% coupon, at yield to maturity of 30%. The hit taken by par values reflects an effort to make the debt payable by keeping coupons reasonable. Note that these numbers also reflect support via additional loans and the renegotiations of existing debt burden.

Refer Exhibit V

Although the focus of this paper is the analysis of a piece of financial data to evaluate PLS methods, the reader may welcome a little bit of information about Greece's case. Its evolution has continued to be troublesome, reaching a breaking point in the midpoint of July 2015. Greece received different financing from the European Union and from the IMF. A first bailout took place in May 2010 (€788 billion pledged by the Eurozone, and €308 billion by the IMF). A second bailout occurred in March 2012 (€344 billion, Eurozone; €288 billion, the IMF). Each time things have become more complicated as participants have needed to keep track of disbursements, roll overs from one bailout into another, changing conditions concerning interest and principal payments, and assessing new financing needs for a potential new bailout to be definitive. The stress also causes divisions concerning different options available and strains the relationship between Eurozone partners (e.g., France-Germany), to the point where some have recommended a temporary time-out of the Eurozone for Greece. The process is also most damaging to the Greek economy, already fragile in the aftermath of the 2008 crisis, and must have taken a toll on the startup segment of its economy.

The durations reflect a downward trend and, in two periods (around May 2010, and May 11) their dynamics seems to anticipate incoming trouble in the series, although they may also indicate the markets' anticipation of difficult European summer periods --note the areas indicated by the oval superimposed shapes in all the series.

Exhibit VI presents the computations for Italy (the third largest economy of the Eurozone with a 16.51% of the 9.487 trillion euros GDP of the 17 countries-Euro area; www.eurostat.com , GDP and main components - Current prices [nama_gdp_c], year 2012 figures). The series does not have as much "drama" as that of Greece, but it does not mean that Italy has not generated its share of concern. All in all, from the period covering the early days of May 2010 to July 2012, Italy has managed to end up maintaining about 100% par values in their 10-year notes by increasing their yields by 179 basis points (1.1794 %), mostly through an increase of 150 basis points in coupon rates. This is also observable in the stability of durations around the 8-year marker. Both the quadratic and the cubic estimate for duration indicate a period of intense instability concerning fixed income.

Refer Exhibit VI

Exhibit VII shows the Spanish case. The dynamics in the time series are less dramatic than that of Greece but indicate a more pronounced instability than the Italian series. Spain is the fourth largest country in the Eurozone (11.06% of the 9.487 trillion euros GDP of the 17 countries-Euro area; www.eurostat.com , GDP and main components - Current prices [nama_gdp_c], year 2012 figures). Spain, like Italy, may be expected to have more mature internal financial markets than those in Greece and Portugal. Unfortunately, the European crisis shows examples of both large and small countries (e.g. Ciprus, Iceland, Ireland, Portugal) overextending themselves in previously unseen manners. In December 2011, a newly elected government earned a considerable majority in the Spanish Parliament, and had several major priorities to confront. One of them was the banking entity Bankia, whose 4.5 billion euros bailout in 2010 had not been enough to get it out of insolvency and which had to become partly nationalized after requesting 19 billion additional euros. The insolvency of the giant Bankia triggered investors' deep concerns regarding other entities. Trouble in banks and thrift institutions (known as "Cajas de Ahorrro") dried out the liquidity of thousands and thousands of real estate developers, while house prices fell nearly 20-30% in some areas, and many homes remained unsold. The cruelest indicator of the crisis, unemployment, shot up to a nearly 30% level, being specially hard on young people looking for their first employment.

Refer Exhibit VII

As in the United States, the Spanish crisis is also associated with real estate. Part of the prosperity during the previous years rested in the construction of new homes, despite the already existing relatively large household home ownership in Spain (about 80%, one of the largest in Europe). Some research indicates Spain may have built more homes each year during the 1997-2006 period than France, Germany and the U.K combined. It is also true that during many years, European retirees looked for the good weather and welcoming atmosphere of coastal Spain, increasing the demand for new homes.

Spain held multilateral negotiations with the IMF, the ECB, and the European Commission through 2010 but did not accept any official bailout of its government debt. This decision offered Spain with some flexibility in meeting targets without the stringency and inflexibilities normally attached to bailouts. The current perception is that the decision worked well worked in that respect. Sovereign debt costs are hovering in slightly above 4% and the debt was on its way towards the 3% of GDP target around late April 2013. Unfortunately, an unforgiving unemployment level of 27.2 has taken hold of Spain to a depression-level and will affect economic policies in the near future.

Exhibit VIII shows the computations for Portugal (the tenth largest economy of the eurozone, 1.74% of the 9.487 trillion euros GDP of the 17 countries-Euro area; www.eurostat.com , GDP and main components - Current prices [nama_gdp_c], year 2012 figures). The duration dynamics show a continuous period of high instability with continual flare-ups as well. The 10-year note started offering a 4.8% coupon in May 2010, and selling nearly at par (98% of nominal), resulting in "normal" durations of 7.5 years. They ended highly intervened in July 2012, offering a lower coupon of 3.85% , and a yield of 9.7%, reflecting a much discounted price of about 66% of the par value.

Insert Exhibit VIII

There is much we have not touched upon in our short framing of each country's case. We have not mentioned the many economic measures that have had a direct influence on the debt indicators such as, for example, the "haircuts" some investors had to take on some countries' debt (e.g., 55.5% on Greek debt following an agreement on February 2012), invitations to swap old debt for new, and/or maturity extensions. Nor have we mentioned any changes in the ratings which, in the case of sovereign debt, are always relative to the political will to tackle the situation.

In addition to the troubled economic sea of the period we have studied, the debt troubles we have seen would have torpedoed and sunk many corporations and would have lessen the importance of durations and other risk indicators. Because of the international support mechanisms, sovereign debt indicators have kept some measure of plausibility and credibility. In sum, it is remarkable that PLS methods do indeed provide both reasonably reliable instantaneous risk indicators and also capture critical clues about their dynamics as well. This may warrant further research, as briefly indicated in the next subsection and in the Appendix.

B.3 Further research

Further research could be implemented along two complementary avenues. One of these is practical. We have focused on the 10-year note rate from some European countries because they seemingly offer the most suitable medium to study the methods we have proposed and the matter in which they may help is of considerable importance. Nevertheless, the methods we have presented in this study could be applied to any economic and financial time series, and their first and second derivatives could be interpreted as representing the speed and acceleration of changes, respectively. For example, Latin American economies have watched with much attention the events affecting the euro because any changes in policies may have important effects on their ability to float debt, and also affect their exchange rates. In 2013, there has been a great deal of scrutiny of the policy moves by the European Central Bank because easing monetary policy in Europe affects these countries directly through their exchange rates and also in triangulation with the dollar. This is the case with the pesos of Mexico, Chile, Colombia, and Argentina, Brazil's real, and the Peruvian sol. The prices of raw materials and commodities, and export and imports for emerging economies, are also ideal candidates to employ PLS methods.

These practical explorations may be carried out in new methodologies such as fuzzy sets. Relationships like those in Figure B of Exhibit IV are similar to the adaptive fuzzy systems used in engineering, see Kosko (1997). Moreover, polynomial least squares are part of the non-parametric toolbox researchers may employ to detect nonlinear dynamics in observed data which, in many difficult cases (e.g., chaotic systems) is carried out by detecting densities, Vialar (2009).

The other avenue of further research is theoretical. As we indicate in the Appendix, polynomial methods are well-grounded in analytical, numerical, and empirical research, and while they do not promise comprehensive, wide-scale modeling, some progress is indeed possible, as Bronshtein and Semendyayev note (1985, p. 729): "Empirical formulae are no laws of nature yet, but can be important steps towards the discovery of theoretically substantiated functional relationships between physical quantities. An almost classical example of it is the discovery of Planck's radiation law on the basis of three empirical formulae."

Concluding Comments

This study started as a pedagogical exploration on how polynomial least squares (PLS) could help in computing fixed income risk indicators. These methods have essential properties that

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 8 August, 2015

suggested their use in the analysis of data undergoing intense changes: 1) require only a few observations, 2) make full use of the information provided by those few observations, 3) offer convenience in adding higher order terms; and 4) the can process most recent, actual observations.

The implementation strategy was twofold. We first explored which polynomial specifications best captured the presumed theoretical properties of the data. Thereafter, we applied those specifications to the time series of sovereign debt for certain European countries undergoing extreme financial stress.

Our research shows that a polynomial least squares specification of third degree allows for computing duration in a reliable manner which, nonetheless, is sensitive enough to reflect changes in the data and in the underlying economic processes. In some cases, PLS methods may anticipate incoming trouble. Our results are encouraging enough to warrant further explorations of dynamic analysis using PLS methods, which already are well-known in engineering, and can be applied using different methodologies and modeling contexts. The biggest appeal of the methods presented is their empirical foundation and applied focus. In sum, paraphrasing a comment made regarding portfolio optimization models, we can conclude the following: in practice, where crude approximations may be better than none, polynomial least squares methods may be found to have pragmatic usefulness.

Appendix

In the text, we have studied the relationship between a second-degree order Taylor expansion and second order polynomial least squares —also known as parabolic least squares. In this appendix we will point out the relationship between the nth-order Taylor expansion and the corresponding nth-degree polynomial least squares, which completes our investigation of the application of PLS to model the relationship between prices and yields.

Hypothesize a functional relationship between a bond's price and its yield to maturity per the prior Equation (1) in the main text. The equation is n-differentiable, which means that we could study its local behavior around a given point with Taylor's expansion, whose general form is:

$$f(x) = f(x_0) + f'(x)\,(x-x_0) + 1/2\, f''(x_0)\,(x-x_0)^2 + \ldots$$

$$\ldots + 1/n\, f^n\,(x_0)\,(x-x_0)^n + R_n \qquad \text{(A1)}$$

R_n, the Lagrangian remainder, collects all the information missed by the previous approximations. This can be found in nearly all textbooks used in introductory quantitative courses —see for example Chiang (1984, p. 258-262), or Bronshtein and Semendyayev (1985, p. 245). Taylor's formula is also a useful educational device, which can be used to illustrate important items in calculus such as the mean-value theorem and Rolle's theorem. In bond analysis, f(x) is usually the bond's price; therefore, $f(x) - f(x_0) = \Delta$price, and $(x-x_0) = \Delta$yield. One rarely needs to calculate the Lagrangian. However, it is appropriate to keep in mind that Taylor's approximation requires a differentiable function of order $(n+1)^{th}$ that converges over the range of

x. See Brooks and Livingston (1992) for a thorough application of higher order Taylor's expansions to fixed income.

Empirically, we could explore the best order through n-degree polynomial least squares, which posit a linear relationship between the price and the m-different powers of the independent variable, even though the relationship between the price and the yield is not itself linear. Polynomial least squares (PLS) are an improvement over Taylor approximations in several ways: a) they can be applied to few observations, b) analytical derivatives need not be calculated,
c) those derivatives are computed numerically, and d) they often provide better approximations to the parameters of interest. The mth degree polynomial equation is:

$$f(x) = a + b\,x + c\,x^2 + d\,x^3 + \ldots + m\,x^m \qquad \text{(A2)}$$

This equation approximates a data set of n observations, $n \geq m + 1$: $[x_1, y_1]$, $[x_3, y_3]$, … $[x_{m+1}, y_{m+1}]$. The best fitting curve has the least square error, which is given by:

$$\Pi = \sum_i [\, y_i - f(x_i)]^2 = \sum_i [y_i - (a + b\,x + c\,x^2 + d\,x^3 + \ldots + m\,x^m)]^2 \qquad \text{(A3)}$$

The "normal" equations used to obtain the parameters —Equations (15) and (18) in the main text—are obtained by taking first order derivatives of Π, and equating the resulting set of simultaneous equations to zero. Finding the parameters a, b, c, … also implies that the polynomial does, in fact, pass through the channel represented by the series of points. In the context of our topic, a vector of market prices could be fitted to some associated observations on certain key rates. The models thus obtained could be evaluated as nested model selection alternatives, see Judge et alia (1985). In practice, low order approximations (quadratic, cubic) may prove most useful because they capture the essential elements of nonlinear behavior. Polynomial least squares appear in introductory books such as Spiegel (1992, p. 267) as a straightforward approach to modeling nonlinear relationships. One can also find material on polynomial approximations in monographs dedicated to linear algebra —Strang (1988, p. 98)— or Barnett (1990, p. 107). These references, however, barely uncover the tip of a very large iceberg: the approximation of empirical functions. Function approximation, smoothing techniques, and interpolation procedures apply numerical techniques to empirical data where polynomials play a major role. There are several important theorems (Chebyshev's, Weierstrass), and techniques (Chebyshev's, Lagrange's Newton's, Gauss, Bessel, Stirling), that relate a set of points to specific approximating/smoothing and interpolating polynomials. A polynomial p(x) is said to interpolate the data set when p(x) = y = f(x), that is, when the curve passes through the data points. Polynomial least squares are interpolating polynomials. Approximating techniques may provide curves that pass in-between or around most data points because we cannot find parameters that replicate the data set. In other cases, we avoid exact data points on purpose. For example, in the stock market some investors use moving average techniques because they smooth out historical price observations. In this study we wanted to stress the adequacy of low-order polynomial least squares to provide estimates of duration and convexity. Some numerical analysis techniques are likely to provide excellent approximations (e.g., Chebyshev's or Newton's polynomials, Stirling's numbers), but they may not provide economically meaningful expressions for duration or convexity. A thorough overview of numerical analysis techniques can be found in Bronshtein and Semendyayev (1985).

Let us add that polynomial least squares seem well-suited to analyze expectational data of a economic and financial nature. The matrix that appears in polynomial least squares —Equations (14) and (17) in the main text— is known as a Vandermonde matrix. Its determinant exists and it is unique as long as the rates are different —two equal rates would result in two identical rows. It can be shown that the determinant is the product of the differences in the set of rates. Again, as long as rates are different, the inverse exists and it is unique. The set of interpolating equations, (14) and (17) in the main text, have only one solution. The "law of one price" is the economic parallelism that comes to mind. The Vandermonde matrix also has important dynamic properties that are used to find an explicit solution of differential equations, Bellman (1970, Chapter 11). Lastly, a system with a Vandermonde matrix is controllable, see Barnett (1990, p. 84).

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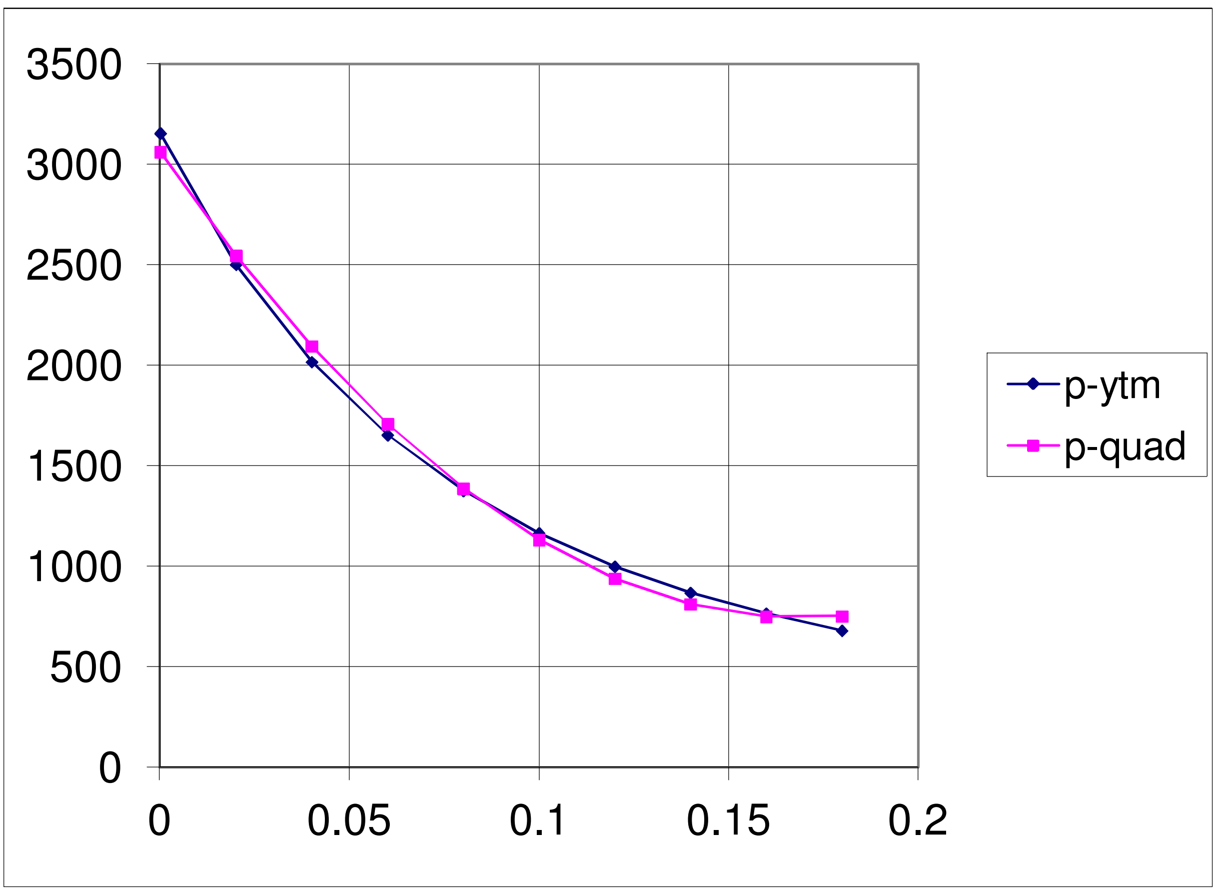
Exhibits

Exhibit I. Research strategy

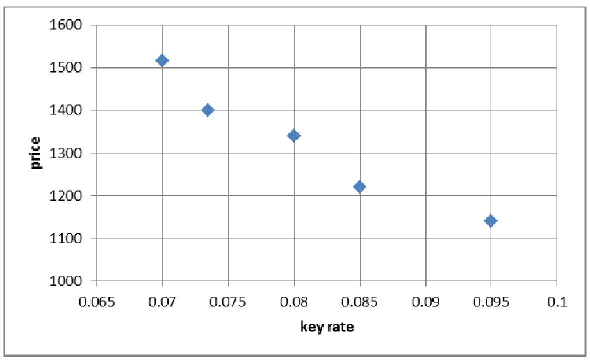
Framework\tools	**Exact**	**Approximate**
Theoretical	Textbook, Abstract research	• Observations • Order of approximation
Empirical	Does not exist	Practical and Applied work

Exhibit II. Price-ytm relationship: quadratic and cubic approximations



				Quad approx to p-ytm		Cubic approx to p-ytm	
ytm	p-ytm	p-quad	p-cubic	error	error^2	error	error^2
0.0001	3156.20	3062.50	3143.61	93.71	8780.66	12.59	158.51
0.0200778	2503.20	2547.05	2520.01	-43.85	1922.84	-16.81	282.64
0.0400556	2018.39	2096.30	2028.70	-77.90	6069.15	-10.31	106.27
0.0600333	1654.44	1710.25	1650.38	-55.81	3114.75	4.06	16.49
0.0800111	1378.03	1388.91	1365.73	-10.88	118.34	12.30	151.22
0.0999889	1165.57	1132.27	1155.45	33.30	1108.80	10.12	102.47
0.1199667	1000.24	940.34	1000.21	59.90	3588.27	0.03	0.00
0.1399444	869.97	813.11	880.71	56.86	3232.62	-10.74	115.35
0.1599222	766.02	750.59	777.63	15.43	238.01	-11.61	134.81
0.1799	682.03	752.77	671.66	-70.75	5005.02	10.37	107.52
			MSE	0.00	3317.85	0.00	1175.28
			Root-MSE		57.60		34.28

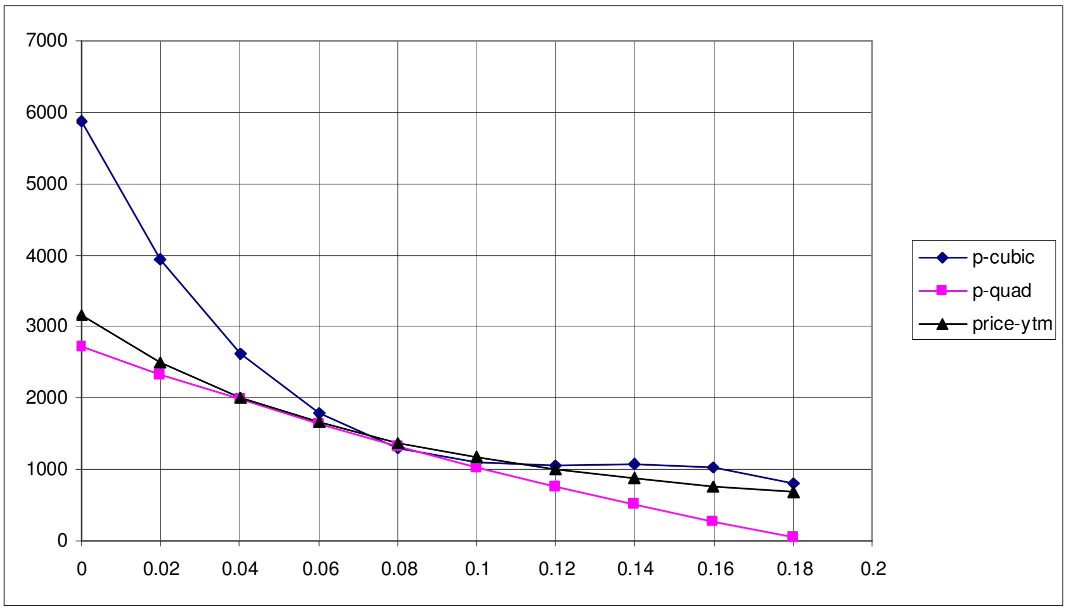
Exhibit III. Empirical price-rate relationships



	estimates	
	quadratic	cubic
a	2712.213	5876.539
b	-19151.4	-112795
c	24008	877405.5
d		-2264403

	Price approximations		
obs	p-quad	p-cubic	p-ytm
1	1489.251	1503.46	1507.262
2	1405.552	1426.927	1460.109
3	1333.749	1308.93	1378.166
4	1276.95	1237.561	1319.739
5	1109.498	1138.121	1213.65
MSE	244.3007	711.8291	1709.86
Root-MSE	15.63012	26.68013	41.35045

obs	rate	market price
1	0.07	1515
2	0.0735	1400
3	0.08	1340
4	0.085	1220
5	0.095	1140



International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 8 August, 2015

Exhibit IV. Empirical price-rate relationships.

Figure A

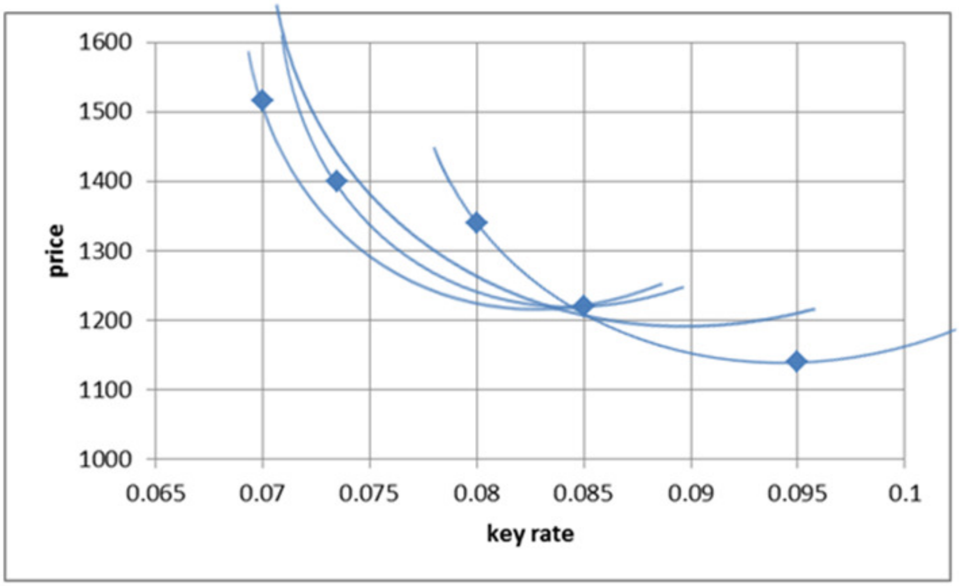


Figure B

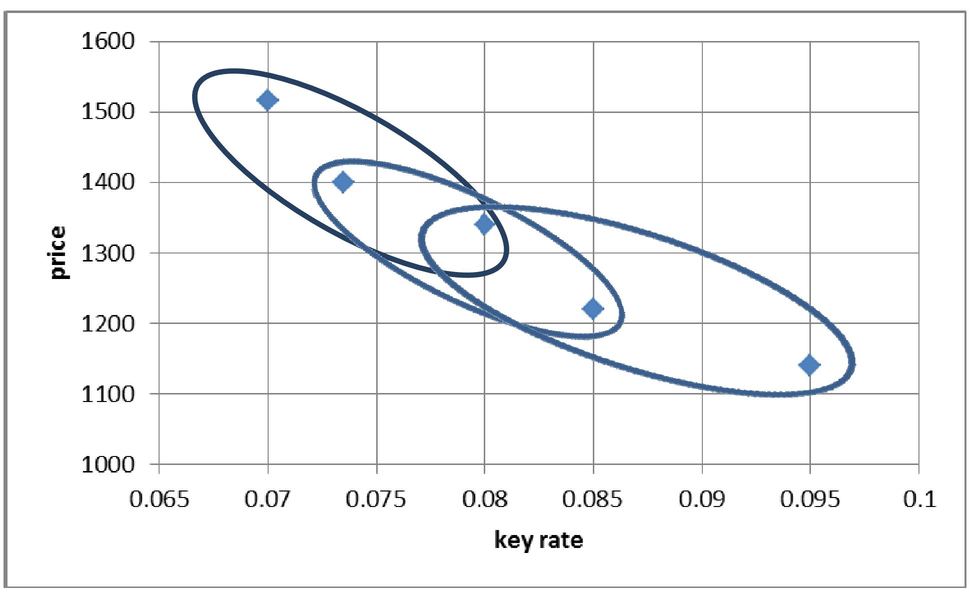


Exhibit V. Greece

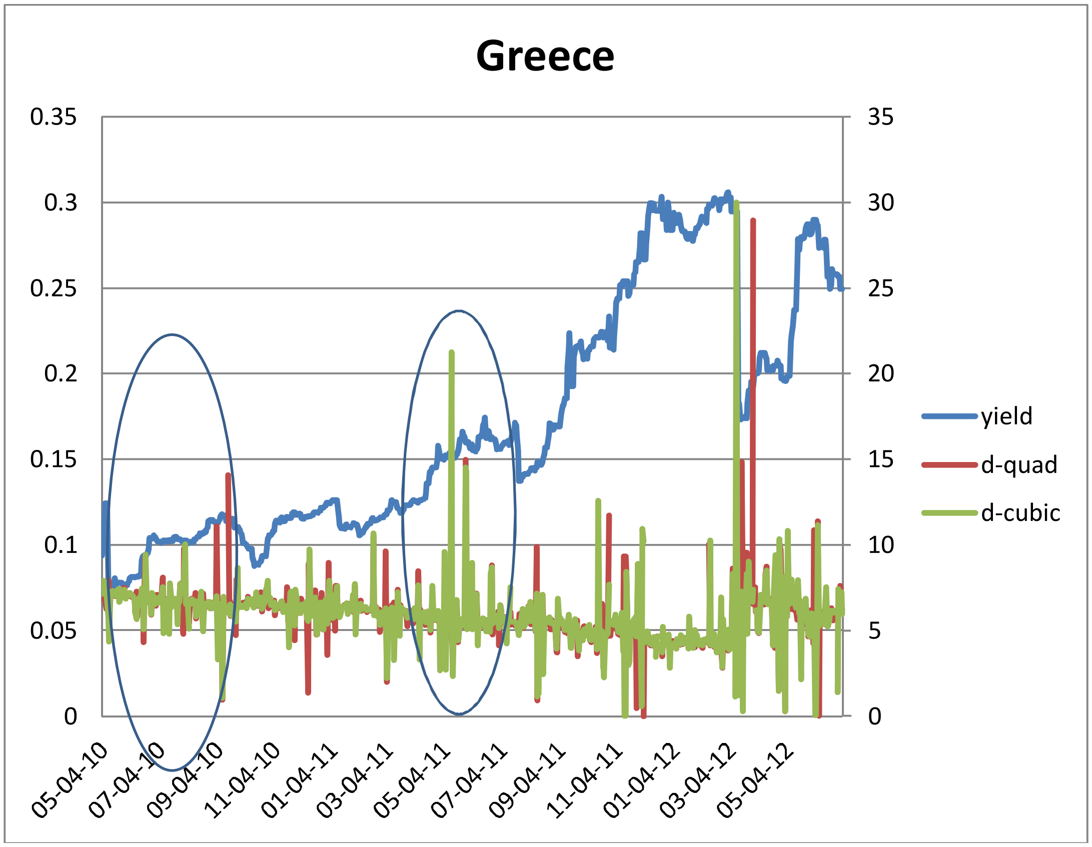
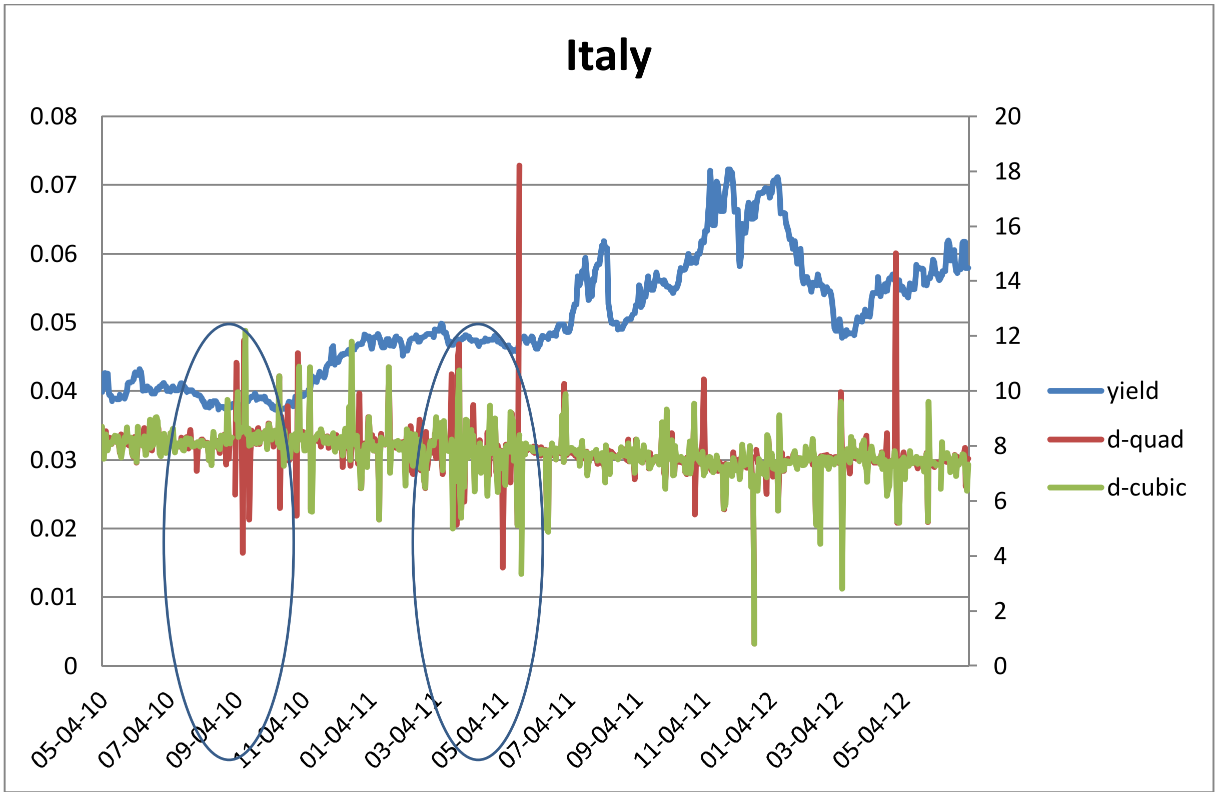


Exhibit VI. Italy



International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 8 August, 2015

Exhibit VII. Spain

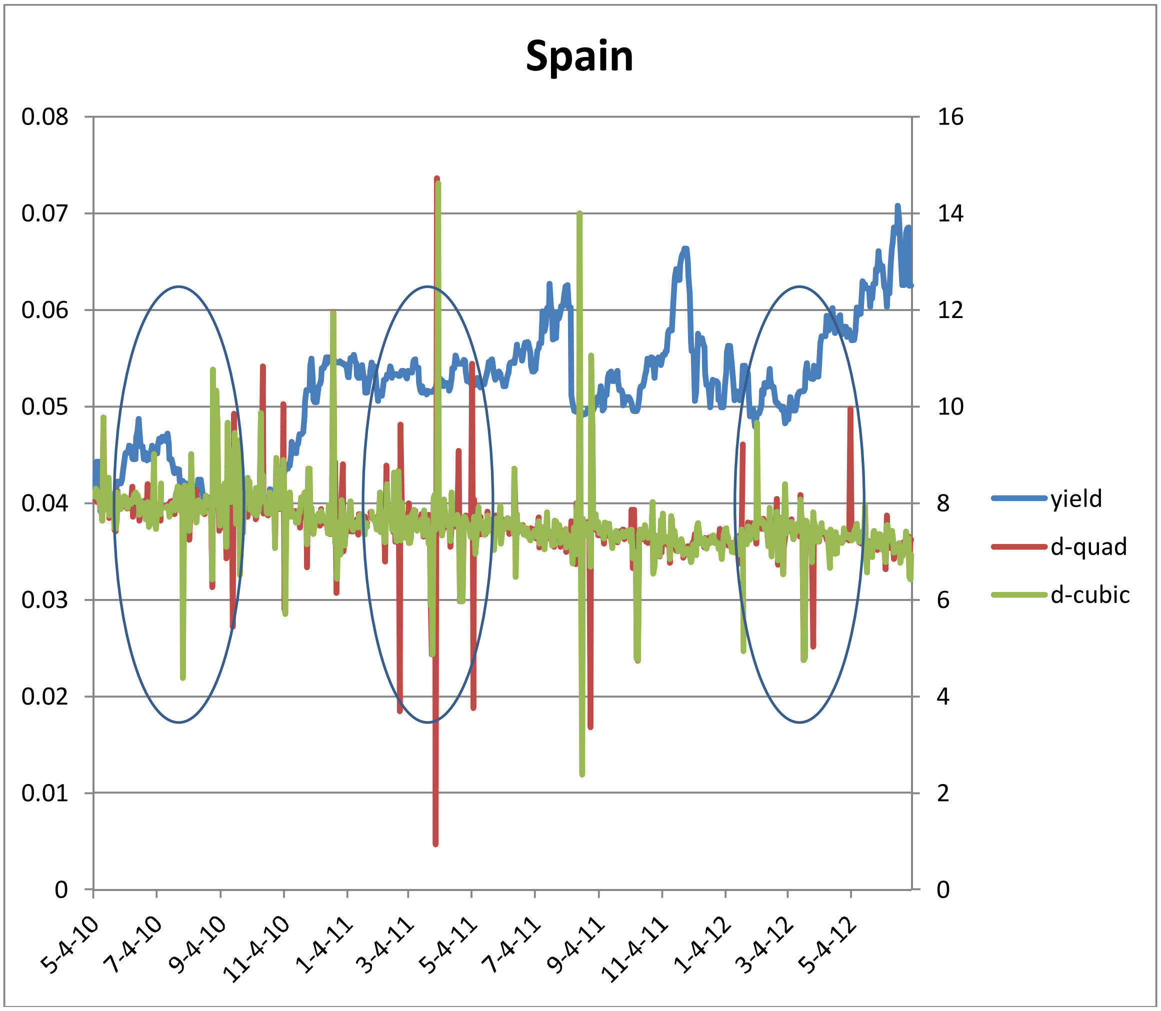
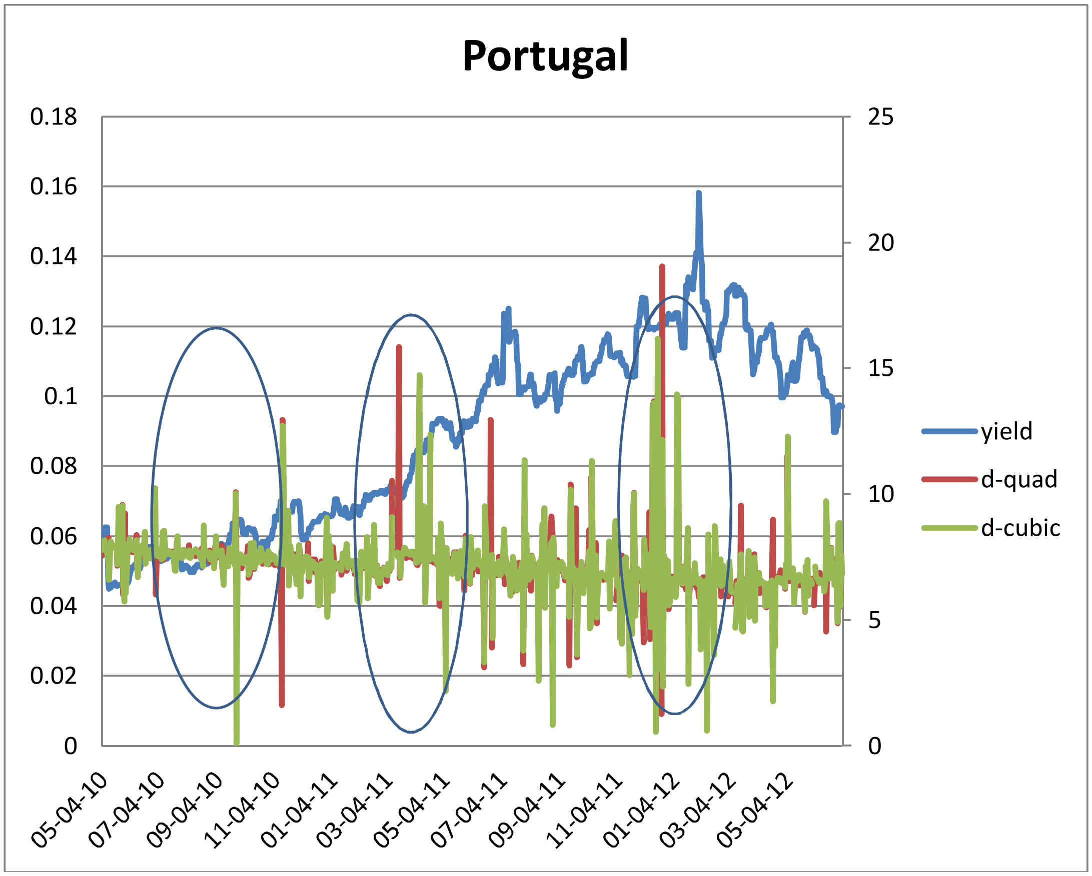


Exhibit VIII. Portugal



Author

Manuel Tarrazo, PhD
Professor of Finance, School of Management, University of San Francisco, 2130 Fulton Street, San Francisco, CA, tarrazom@usfca.edu