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Optimal Dynamic Hedging of Cliquets

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Abstract. Analyzed here is a Cliquet put option (ratchet put option) defined as a resettable strike put with a payout triggered by the reference asset falling below a specified fraction of its value at a prior look-back date. The hedging strategy that minimizes P&L volatility over discrete hedging intervals is assessed. Examples are provided for an asset exhibiting jumpy returns (kurtosis > 3) and temporal correlation between the squared residual returns. The limited liquidity of the asset limits the discrete hedging frequency. Each of the realities of discrete hedging intervals and fat-tailed asset return distributions render the attempted replication imperfect. A residual risk dependent premium is added to the average cost of attempted replication (i.e., average *hedging* cost) based on a target expected return on risk capital. By comparing the P&L distribution of a derivative seller-hedger with that of a *delta-one trader* holding a long position in the underlying asset, relative-value based bounds on pricing of vanilla options and Cliquets are presented.

Keywords: *Gap-risk, Cliquet, Crash-Cliquet, Kurtosis, Hedging, Residual risk, Option traders P&L*

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1. Introduction

The P&L of a seller and hedger of a Cliquet contract on an asset with limited liquidity and with jumpy returns is analyzed here. The hedge ratio that minimizes P&L volatility, the average hedging cost, and the hedge slippage probability distribution are assessed by applying the Optimal Hedge Monte-Carlo (OHMC) methodology developed by Bouchaud & Potters [2003]. The computed probability distribution of the option-seller-hedger's P&L reflects the stochastic characteristics of the asset, the hedging strategy, and the Cliquet contract. We determine the risk-premium that needs to be added to the *average hedging cost* to render the risk-return of the *derivatives trader* (that sells and hedges the option) to be no worse than a *delta-one trader* who is long the underlying asset.

Cliquets in the equity markets are often in the form of out-of-the money put Cliquets that are used to protect the holder from a market crash scenario (i.e., crash Cliquet, gap risk Cliquet). Most *gap-risk* Cliquets are defined as forward starting put spreads (e.g., 85-75 strike Cliquet put spread). This structure is similar to a tranche of a market value CDO (with attachment points of 15% and 25%). Hedging the mezzanine tranche of such a CDO involves trading the underlying assets to protect against *gap risk*. These structures share a common feature: sudden large moves of the underlying asset can cause economic loss. The OHMC methodology proposed below will explicitly include such moves through a process that exhibits excess kurtosis and results in credit-type loss mechanisms.

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Petrelli et al [2006] analyzed optimal static hedging of multi-name credit derivatives (i.e., synthetic CDO tranches). Kapoor et al [2003] employed a GARCH(1,1) model in the OHMC framework to examine the risk return characteristics of two-tranche structures supported by a volatile asset. In that work the underlying asset values were reported monthly and the hedging interval was also monthly. The errors incurred in replicating the senior tranche were compared to that incurred in replicating the junior tranche, in addition to computing the average cost of attempted replication. The impact of knockout and running premium for a Cliquet contract results in a wealth change formulation that is quite similar to that for a CDS swaption problem analyzed in Zhang et al [2006]. This work builds up on the formulations in Kapoor et al [2003] and Zhang et al [2006] and applies it to a Cliquet contract, and further analyzes residual risks and return on risk capital. The ultimate goals of our series of works on optimal hedging are to be able to handle hedging, attempted replication, and assess residual risks of multi-asset options - be they credit type assets or equity type assets. This is also a prerequisite to developing a satisfactory *valuation* model for multi-name credit derivatives. The reader is referred to Petrelli et al [2007] for documentation of dynamic-hedge performance of CDO tranches, and to Laurent et al [2008] for a direct study of replication of CDO tranches.

In the OHMC approach the asset underlying the derivative is simulated based on a model that seeks to capture its real-world characteristics. The hedge ratio and pricing functions are sought at every time step to keep the hedged derivative position as flat as possible between successive hedging intervals. The numerical solution for hedging starts at the time-step prior to the option expiry. The hedge ratio and pricing functions are a solution to the variational-calculus problem of minimizing a statistical hedging error measure between two time-steps while keeping the trading book flat on the average. Thus the OHMC methodology puts itself in the shoes of a derivatives trader attempting to replicate the option payoff. Like the derivatives trader, the OHMC methodology is concerned with residual risk accompanying any hedging strategy in the *real-world*. OHMC seeks to deliver to the derivatives trader information on the average hedging costs and the residual risks. By comparing the average cost of hedging and the residual risks with the amount of money someone is willing to pay the trader for that option, an opinion on the attractiveness of the trade can be developed.

OHMC Versus "Risk-Neutral" Expectations

The optimal hedging methodology adopted here follows the approach taken by Bouchaud and co-workers: see Bouchaud and Potters [2003] for an introduction to OHMC. Other foundational studies of optimal hedging include Schweizer [1995], Laurent & Pham [1999], and Potters et al [2001]. All of these works are focused on the cost of option replication by analyzing dynamic hedging explicitly and as a pre-requisite to *valuation*.

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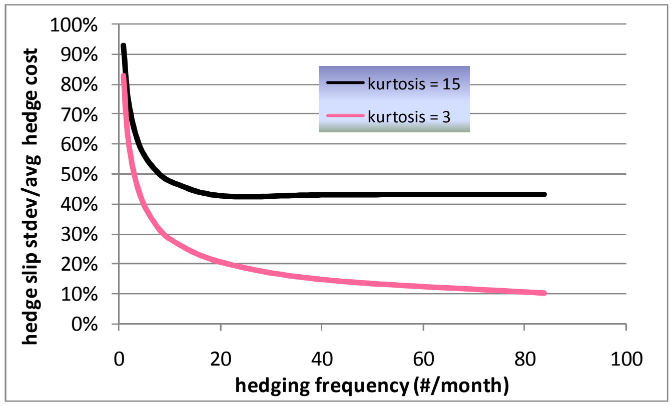
The works on optimal hedging mentioned above and the approach pursued here are easily distinguished from the formal *risk neutral* valuation approaches insofar as follows:

- OHMC addresses the mechanics of hedging, the average hedging costs and the hedge slippage distribution, and establishes the theoretical reasons for lack of perfect replication. Taking expectations under a *de-trended* underlying process does not directly address hedging and replication errors (in *risk-neutral* modeling hedging errors are assumed to be zero under *ideal conditions*).

- OHMC is applicable to general stochastic descriptions of the underlying. Risk-neutral valuation - with demonstrable theoretical perfect replication - is limited to very specific descriptions of underlyings that are not empirically observed in even the most vanilla and liquid financial instruments, let alone exotic underlyings. For example, the daily return kurtosis of US large cap stocks is on the average ~ 20: see Bouchaud & Potters [2003]. The kurtosis of the daily and monthly returns of the S&P500 total return index is many multiples of 3 (the kurtosis of Geometric Brownian Motion).

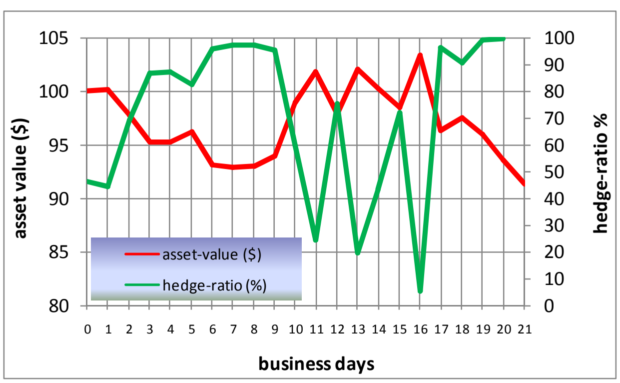
The residual risk associated with any attempted replication strategy is of paramount importance to a derivatives trader trying to carve the risk-return profile of a trading book, and making the binary decision of selling an option at a given price in the first place. OHMC is focused on hedging strategies, their expected costs, and residual risks.

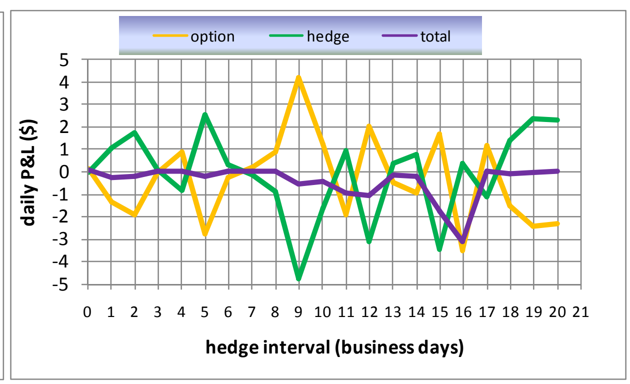
The central idea of derivative replication is establishing costs and trading strategies for *eliminating* risk. However, valuation modeling has come to limit itself to taking expectations of option payoffs under a *de-trended* underlying process – without establishing the mechanics to achieve replication or estimates of the residual risks. Such formal *risk neutral* models (that do not establish replication but presume it to be theoretically possible) are generally *fit* to market prices – without offering any analysis of hedging and its limitations. The *risk-neutral* label seems to be earned merely by taking LIBOR discounted averages of option payoff evaluated using a *de-trended* description of the underlying relative to the cost of carry (which in recent environments has itself jumped around!). Such formal *valuation* models do not differentiate options based on the relative sizes of replication errors endemic to the option contract and the underlying process. While such formalism based *valuation modeling* is taking hold in accounting practices, it is largely an exercise of fitting model parameters to observed derivative prices. Such formal risk neutral models do not directly help understand risk-return characteristics or hedge performance of vanilla options or exotics. The main role of such *risk neutral* valuation models seems to be *facilitating* upfront P&L for exotics by employing parameters fitted to vanilla options, with the presumption that a delta to the underlying and vanilla options can *perfectly* replicate an exotic.



(a)

Figure 1a. Dependence of hedging error on hedging frequency and return kurtosis for a 1 month at the money put option (see Example 1 in **Appendix-II** for further details). The hedge error measure displayed here is the standard deviation of hedging error divided by the average hedging cost. The black line shows results for a stylized asset (stylized-asset 3 defined in **section 4**) with a return kurtosis of 15. The rose-pink-line shows results for the same asset, but without any excess kurtosis, i.e., with a kurtosis of 3 which corresponds to a Geometric Brownian Motion (GBM) rendition of the asset.





(b) **(c)**

Figure 1b&c. Sample path behavior of OHMC analysis with daily hedging for a 1 month at the money put option for a stylized asset with return kurtosis equal to 15 (see Example 1 in **Appendix-II** for further details). The evolution of the asset-behavior and the hedge ratio is shown in **(b)**. The daily P&L is shown in **(c)** with contributions from the option position and the hedge position (discounted to start of hedging interval). The P&L plot does not show the risk premium the option seller will add to the average hedging cost to get compensated for the residual P&L risk. The sample path shown here corresponds to the 1 year 99.9% confidence level equivalent total wealth change over the options life.

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Reality Versus Perfect Replication & Unique Price

Minor jumpiness of returns (excess kurtosis) rules out perfect hedging even in the continuous hedging limit for vanilla options – as depicted in **Figure 1** - based on an example of an *at-the-money* put further detailed in **Appendix-II**. This fact is brazenly ignored by the mainstream valuation modeling but experienced by the derivative trader herself. The unsophisticated model user can fall into the convenient trap of *believing* that the only consideration in a derivatives trade is how *correct* is the volatility surface (or parameters of stochastic volatility models) in valuing a derivative and calculating the standard greeks (delta, vega, gamma). Accounting departments (also called *Product Control*) further reinforce this risk-free replication *belief* based modeling regime – by embracing the *unique derivative price* found after calibrating parameters to observed vanilla prices. This ignores the bid-offer of prices in vanilla derivatives and what they may reflect about the residual risks in attempting to replicate a vanilla option. At greater peril, for exotics, this ignores the fact that the residual risks in replicating an exotic may be quite different than the vanilla option and *a-priori* one should have no expectation that calibration to some presumed *mid* price of vanillas results in a replicating strategy for the exotic.

Risk management departments (also called *Risk Control*) are charged with feeding the pricing model sensitivity outputs into a VaR model, and generally defer P&L deliberations to accounting. Accounting departments in turn often defer to a *valuation model* that purports itself to be *risk-neutral* and does not *a-priori* communicate estimates of hedging errors. VaR models vary in terms of granularity of market risk factors (index, vs. single name, etc) and their capabilities in making assessments of P&L with sensitivities and-or a complete revaluation. Due to its simplifications, often VaR is not reported at a trade level. With the valuation model based on presuming perfect replication, and the VaR model typically broad-brushed and not focused on the quirks of exotics, it is quite possible that trades are executed – possibly strongly motivated by the upfront P&L or significant carry – yet without any careful assessment of risks. Therefore a proliferation of *risk* neutral valuation models has not generally been accompanied by an improvement in risk management. In-fact, valuation models that purport to be *risk*-neutral and do not advertise irreducible hedging errors, perpetuate the incorrect belief that a delta-hedged position is close to risk free and actually aid and abet the taking of *un-sized risks*, despite the seeming oversight of *valuation* modeling, risk-control, and product control.

The diligent trader/risk manager will typically get to understand the exotic derivative over time and may know how its risk sensitivities are different enough from the vanilla derivative such that calibration of model with vanillas does not ensure a plausible pricing of risk-premiums that are endemic to attempting to replicate the exotic. The “risk-neutral” valuation models complete silence on hedge performance (as a part of valuation) renders them of little value in developing an exotics trading strategy with clearly documented risk-return tradeoffs.

The amount of compensation that a market agent that sells an option (and attempts to replicate) demands for unhedgable risks is for that agent to opine on and for trading counterparties to be the ultimate arbiter of. The amount of risk premiums the market will bear will depend on demand and supply, market sentiments, and the extent to which the residual risks are diversifiable in a practical trading book. Replication and diversification are at the heart of making derivative trading decisions, and derivative *valuation models* that are silent about even theoretically unhedgable risks are of little value in guiding trading and risk management in expressing a risk preference. The OHMC approach has the potential to more directly tie together valuation modeling, hedge performance analysis, and risk management, by relating the average hedging costs and hedge slippage distribution to the distribution of the underlying as a precursor to valuation, and therefore prior to the stage where upfront P&L or carry motivations are entrenched. The purpose of this article is to demonstrate the practical feasibility and the utility of the optimal hedging approach for Cliquet contracts. It will be demonstrated that the OHMC framework can be utilized as a consistent hedging, pricing, risk-management engine that can serve the needs of traders, risk managers, and product control (see **Table 1**).

Cliquet contracts are traded for liquid public market assets as well as more customized assets that could represent a trading strategy itself, in the form of either a rule based strategy, or a hedge fund, or an associated index. The values of such customized assets are often reported at a much lower frequency than the liquid public market assets, and the liquidity time interval over which any hedger can adjust the hedging portfolio can be even larger than the interval over which asset-values are reported. The hedging of Cliquet contracts for such imperfectly liquid assets is one of the focus of this work, hence the explicit treatment of hedging frequency. The bespoke baskets motivating this work are more widely held as long positions, therefore hedging requiring going long the underlying bespoke basket may be easier than going short. However it is possible to go short by total return swaps with counterparties that want to go long the bespoke basket. In this work we invoke an ability to go long as well as short the asset underlying the Cliquet contract, with limited frequency of adjusting the hedge due to contractual limitations on redemptions of customized baskets.

Characteristics	Risk-Neutral Approach	OHMC
Hedging & replication	i. Assumes perfect replication is theoretically possible and residual risks are non-existent ii. Continuous/instantaneous hedging with no transaction costs iii. Naïve deltas assuming (i)and (ii) without any reference to real-world risks	i. Seeks to minimize hedging error and assesses real-world residual risks that are generally found to be significant compared to *average* hedging costs ii. Can addresses hedging frequency and transaction costs iii. Produces hedge ratios that minimize desired hedging error measure (e.g., P&L volatility or expected shortfall)
Underlying description	i. De-trended *martingale* process only	i. Independent of underlying process (i.e., applicable to non-markov, fat tails, jumps etc)
Risk management needs	i. Valuation model based on risk-free replication assumption – no direct risk metric output from model ii. Sensitivity of valuation models can be used to evaluate risk measures external to *valuation model* (VaR, expected shortfall, etc) iii. Loss scenarios can be assessed by perturbing valuation model inputs	i. Scenario evaluation/back-testing is performed as a part of assessing cost of hedging and hedge slippage measures which drive *valuation* ii. Risk measures (VaR and expected shortfall) are collateral model output en-route to assessing hedging strategy

Table 1. Comparison of Risk Neutral Approach with Optimal Hedge Monte-Carlo (OHMC)

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Organization

The reader completely new to optimal hedging analysis is recommended to start with **Appendix-II** which introduces it in the context of vanilla options and presents sample calculations on hedging error, average hedging costs and hedge ratios. The reader with background on optimal hedging analysis should jump right into the Cliquet formulation, and **Appendix-II** is then best read prior to **section 5** of the main text.

The mechanics of the Cliquet contract are presented in **section 2**, including a formulation of a Cliquet option seller's P&L. **Section 3** presents the OHMC analysis of the P&L of the Cliquet option seller – hedger, and the approach to finding the optimal hedge ratios and the average hedging cost as well as residual risks. Also presented in **section 3** are relative value and risk capital measures that utilize the assessments of average hedging costs and residual risks and provides guidance on *valuation*. **Appendix-I** and **III** present details of the OHMC implementation for Cliquet contracts. The OHMC method to analyze Cliquets pursued here does not hinge on any special or convenient stochastic description of the asset. Rather, it assesses the average hedging cost and deviations around those averages, given any description of the asset and Cliquet contract parameters. For the purpose of presenting specific examples we use a GARCH(1,1) description of the asset and employ three stylized asset descriptions. **Section 4** presents the GARCH(1,1) description and a method of moments approach that can be used to calibrate its parameters to data. **Section 5** presents some specific examples. A discussion of this work and concluding remarks are presented in **section 6**.

2. Cliquet Contract

The basic Cliquet put option contract consists of a series of forward starting European puts. On pre-specified dates separated by the time-interval between checking payout trigger conditions, τ_{roll}, the value of the reference asset is compared to its value one look-back interval, $\tau_{look\text{-}back}$, earlier. If the asset has fallen beneath a specified fraction, called strike K_1, of its value one look-back interval earlier, then a payout to the option purchaser is made by the option seller. That payout is the fractional amount by which the asset has fallen below the strike K_1 multiplied by a reference notional value – with possibly a maximum payout fraction established by a lower strike $K_2 < K_1$ (sometimes called a "bear-spread"). In the knock-out variant of the Cliquet contract, the contract terminates after one payout event. In return for the possible payout, the option purchaser pays the option seller a running premium and/or possibly an upfront payment. The costs and efficacy of hedging such Cliquet contracts are analyzed here in the framework of minimizing P&L volatility in between hedging intervals. There can be many other variants of the Cliquet contract - the framework developed here can handle any path-dependent derivative.

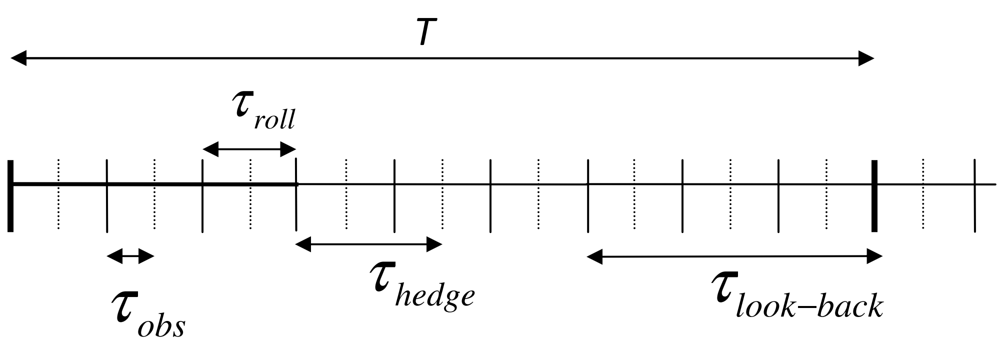

Figure 2. Schematic of time-scales pertinent to the Cliquet contract. The option tenor is denoted by T. The most granular time-interval over which asset value observations are available is denoted by τ_{obs}. The hedging interval is denoted by τ_{hedge}. The look-back interval over which a decline in asset value triggers payoff and a termination of the contract is denoted by $\tau_{look-back}$. The time-intervals over which the drop in asset value is checked is denoted by τ_{roll}.

One of the motivations for this study are Cliquets on assets with limited ability to rebalance hedges, due to contractual limitations on redemptions, and where the value of the underlying assets can be reported less frequently than a typical publicly traded stock or bond or CDS contract. The redemption time-interval and the asset value reporting interval are two time scales that are characteristics of the underlying asset are a constraint on the Cliquet option trader.

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Additionally, the frequency of checking the payout condition and the look-back interval (over-which a decline of asset value triggers payout) are time-scales pertinent to a Cliquet trading strategy. These time-scales are depicted in **Figure 2**. Devising a hedging strategy that is cognizant of these five time-scales and the asset characteristics (including its volatility clustering time-scale introduced later) is our purpose.

A-priori we know that the replication cannot be perfect, unless the description of the underlying asset is contrived. We illustrated that for vanilla options in this paper (**Figures 1** and **Appendix-II**). One of our main goals is to demonstrate the utility of elucidating the residual risks while *attempting* to replicate as much as possible. By combining the average hedging costs and the residual risks, we seek to develop a framework for relative value metrics that will drive the prices.

Cliquet Option Sellers P&L

The terminology and symbols needed to specify a Cliquet contract and a protection seller's P&L are enumerated here.

Contract trigger condition/look-back dates: $\left\{\hat{t}_0 = 0, \hat{t}_1, \hat{t}_2, ..., \hat{t}_{N-1} = T\right\}$

Hedging interval: $\left(t_k, t_{k+1}\right]$

Running premium rate: η

Cliquet value: $C(t)$

Payout trigger time: $\hat{t}_{i*} = \min \hat{t}_i \ni \dfrac{s\left(\hat{t}_i\right)}{s\left(\hat{t}_{i-nlb}\right)} < K_1$

$$i \in \left\{nlb, nlb + 1, ..., N - 1\right\}$$

nlb is the number of observation intervals in a look-back period

Reference notional: Ψ

Payout amount: $P\left(\hat{t}_{i*}\right) = \Psi\left[K_1 - \max\left\{\dfrac{s\left(\hat{t}_{i*}\right)}{s\left(\hat{t}_{i*-nlb}\right)}, K_2\right\}\right]$

Trigger indicator: $I\left(t_k, t_{k+1}\right] = \begin{cases} 1 & \text{payout triggers at } \hat{t}_{i*} \in \left(t_k, t_{k+1}\right] \\ 0 & \text{no payout trigger over } \left(t_k, t_{k+1}\right] \end{cases}$

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Risk-free discount factor: $df(\tau, t)$

Premium accrual time:

$$\chi(t_k, t_{k+1}) = \int_{t_k}^{t^*} df(t_k, \tau) d\tau; \quad t^* = \begin{Bmatrix} \hat{t}_{i^*} & \text{if } I(t_k, t_{k+1}] = 1 \\ t_{k+1} & \text{otherwise} \end{Bmatrix}$$

Discounted payout:

$$\omega(t_k, t_{k+1}] = I(t_k, t_{k+1}] P(\hat{t}_{i^*}) df(t_k, \hat{t}_{i^*})$$

P&L of sell Cliquet protection position between time t_k and t_{k+1}(discounted to t_k):

$$\Delta W_{t_k}^{cliquet}(t_k, t_{k+1}) = C(t_k) + \eta \Psi \chi(t_k, t_{k+1}) - \omega(t_k, t_{k+1}] - \{1 - I(t_k, t_{k+1}]\} C(t_{k+1}) df(t_k, t_{k+1}) \quad (1)$$

The P&L of the sell Cliquet position written above is for contracts with possibly a combination of a running premium and an upfront payment. This specification of Cliquet P&L is used later to propagate the optimal hedging solution from the Cliquet expiry to the initial time step. A similar formulation can be made for any other exotic Cliquets. We focus on the one-touch knockout put variant of the Cliquet.

Hedging the Sell Cliquet Protection Position

In addition to the variables that impact the P&L on the sell Cliquet protection position, the other main object of interest for the Cliquet trader is $\Phi(t_k)$, the amount of asset to hold at time step t_k to hedge the Cliquet position. The P&L generated by the hedge is determined by the change in asset values, the carry costs for owning the economics of the *hedge*, and the discount rates:

$$\Delta W_{t_k}^{hedge}(t_k, t_{k+1}) = \Phi(t_k) \left[s(t_{k+1}) - \frac{s(t_k)}{DF(t_k, t_{k+1})} \right] df(t_k, t_{k+1}) \quad (2)$$

To account for different carry costs (i.e., different funding rates) and possibly dividends or subscription fees associated with the asset, we employ a funding discount factor, $DF(t_k, t_{k+1})$, which is possibly distinct from the risk-free discount factor $df(t_k, t_{k+1})$. Long dated derivative contracts with relatively large hedge ratios can be quite sensitive to the funding rates of the option seller-hedger. For such contracts, the prevailing funding rates of the different market players can be as important as the differences in perceptions about the randomness of the asset

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returns in determining the demand and supply in option markets. In reality the funding rates can also be random, and in some situations can be a dominant determinant of the option cost. While the formulation made here can handle random funding rates and discount rates, we only focus on hedging the impact of the asset value, and invoke constant funding and discount rates for the example calculations made here.

3. Optimal Hedge Monte-Carlo Formulation for Cliquet

Cliquet Seller-Hedger's P&L

We change slightly the notation of the Cliquet sellers-hedgers P&L (**(1) & (2)**) to develop the notation of the OHMC algorithm. The contract value at time t_k, $C(t_k)$, is denoted as a function of the spot at time step k: i.e, as $C_k(s_k)$. The hedge amount at time t_k, $\Phi(t_k)$, is denoted as a function of the spot at time step k, s_k: i.e., $\Phi_k(s_k)$. This notation emphasizes that the value and hedge amount are viewed as time dependent functions of the spot asset value.

$$\Delta W_{t_k}(t_k,t_{k+1}) = \Delta W_{t_k}^{cliquet}(t_k,t_{k+1}) + \Delta W_{t_k}^{hedge}(t_k,t_{k+1}) =$$

$$\begin{aligned} C_k(s_k) + \eta\Psi\chi(t_k,t_{k+1}) - \omega(t_k,t_{k+1}] - (1-I(t_k,t_{k+1}])C_{k+1}(s_{k+1})df(t_k,t_{k+1}) \\ + \Phi_k(s_k)[s_{k+1} - s_k / DF(t_k,t_{k+1})]df(t_k,t_{k+1}) \end{aligned} \tag{3}$$

To keep the notation compact and yet general we rewrite the above as

$$\begin{aligned} \Delta W_{t_k}(t_k,t_{k+1}) &= C_k(s_k) - G_k + \Phi_k(s_k)H_k \\ G_k &= (1-I(t_k,t_{k+1}])C_{k+1}(s_{k+1})df(t_k,t_{k+1}) + \omega(t_k,t_{k+1}] - \eta\Psi\chi(t_k,t_{k+1}) \\ H_k &= (s_{k+1} - s_k / DF(t_k,t_{k+1}))df(t_k,t_{k+1}) \end{aligned} \tag{4}$$

OHMC Problem

In the OHMC approach a MC simulation of asset evolution is used to evaluate G_k and H_k for every random realization. Based on that, all terms of the wealth balance **(4)** can be directly computed, other than the yet unknown deterministic functions of value and hedge ratio $C_k(s_k)$ and $\Phi_k(s_k)$. These two functions are found by imposing a constraint of zero average change in wealth and minimum wealth change variance:

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Find $C_k(s_k)$ and $\Phi_k(s_k)$ so that

$$E[\Delta W_{t_k}(t_k, t_{k+1})] = 0 \tag{5}$$

$$\text{minimize} \quad \sigma^2_{\Delta W_{t_k}(t_k,t_{k+1})} = E\left[\left(\Delta W_{t_k}(t_k, t_{k+1}) - \overline{\Delta W_{t_k}(t_k, t_{k+1})}\right)^2\right] \tag{6}$$

Appendix-I & **III** describes the algorithm to determine the unknown functions $C_k(s_k)$ and $\Phi_k(s_k)$ to satisfy **(5)** and **(6)** given a simulated MC ensemble of s_k that also provide G_k, and H_k.

Residual Risks While Attempting to Replicate

Each of the realities of fat return tails (kurtosis > 3) and discrete hedging individually rule out perfect replication (see **Figures 1** and **Appendix-II**). Certainly the combination of fat-tails and discrete hedging render the residual risk to be of direct interest to someone charged with managing the risk-return profile of a Cliquet trading book. As mentioned in the introductory sections, we think that *valuation modeling* should not be divorced from assessment of residual risks inherent to any attempted replication strategy. OHMC provides a readily implementable avenue to fix the schism created by *formal* risk neutral models that do not address replication-hedging explicitly.

The OHMC hedging time-grid is specified as

$$\{t_0 = 0, t_1, t_2, \ldots.., t_k, t_{k+1}, \ldots., t_{K-1} = T\}$$

As a part of the OHMC algorithm we looked at the P&L between time step k and k+1 discounted to time step k; i.e.,

$$\Delta W_{t_k}(t_k, t_{k+1})$$

The total change in wealth discounted to t_0 is then given by

$$\Delta W_0(0,T) = \sum_{k=0}^{K-2} \Delta W_{t_k}(t_k, t_{k+1}) df(0, t_k)$$

The cumulative P&L from trade initiation to t_k (present valued to time t_k) follows

$$\Delta W_{t_k}(0, t_k) = \sum_{j=0}^{k-1} \Delta W_{t_j}(t_j, t_{j+1}) \left[df(t_j, t_k)\right]^{-1}$$

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The total change in wealth present valued to trade execution time is particularly important because it provides a metric that is directly pertinent at inception. Imperfect replication can/will/should make a Cliquet protection seller ask for a greater spread than that which simply results in a zero average change in wealth. We will examine the distribution of the total change in wealth from the point of view of solvency and associated risk capital, as discussed in the next sub-section. To translate the total hedge slippage into an error term around the fair spread, we normalize total P&L by the product of payout reference notional and average premium payment time:

$$\frac{\Delta W_0(0,T)}{\Psi\,\overline{\chi(0,T)}} \tag{7}$$

The hedge slippage measure in **(7)** is pertinent to judge the risk-return of the Cliquet trade from inception to finish – by summing all the hedge slippage and translating it into a running spread by the normalization in **(7)**. In addition to quantifying the risk-return over the transaction, a trader also wants to control the P&L volatility over smaller time-intervals – say over the hedging interval. We employ both time aggregated hedge slippage and local hedge slippage in assessing risk capital, as described in the next section.

Much of formal risk neutral valuation modeling literature altogether ignores the question of residual risks of replication attempts – invoking the formalism that as long as expectations are being taken under a de-trended underlying, somehow replicating strategies exists in a *risk-neutral world*. Such *risk-neutral* expectations are often taken under descriptions of the underlying that are easily shown to thwart perfect replication for even simple derivative contracts (jump-diffusion, GARCH(1,1), etc). Formal risk neutral expectations are also purported to provide a *valuation* model for contracts for which a replicating strategy is hard to conceive (e.g., CDO tranches whose payouts occur only when jumps-to-default occur). Such valuation models are mainly a parameter fitting exercise and are lacking the information needed by the person charged with *attempting* to replicate the derivative payoff or responsible for trading-risk management.

The issue of residual risks in an attempted replication strategy is operationally often relegated as a risk management topic, whereas valuation modeling focuses on absolutes of *arbitrage-free pricing* and is performed by individuals who are not responsible for managing trading positions and their risks. Therefore hedging analysis has become distant form *valuation modeling*, and one often hears of models for valuation of derivatives that are not models for analyzing hedging! This divorcing of valuation modeling from hedge performance analysis and risk management can account for the poor state of affairs in all these departments, and their marginal role in helping making informed trading decisions. Consequently, poor risk management and lack of understanding of risk return profiles of derivative trading books has often been concomitant with the proliferation of derivative *valuation models*.

Risk Capital

The losses incurred by a derivative trading book can jeopardize the solvency of a financial institution – or certainly the job of a trader or the existence of a trading desk or that of a hedge fund. While ultimately a firm may be interested in its global risk profile, losses at any sub-unit that are disproportionately larger than its size indicate that either extreme odds have been realized and/or that the institution does not understand and can't control the risks of its parts. Reputational damage resulting from financial losses in a subset of a firm can have a detrimental effect on the firm at a global level that go beyond the immediate financial risks. Also, if a clear methodology of understanding risk-return is not expounded at a trade level or a trading desk level, it is unlikely (and dangerous to assume) that risks are understood at a global portfolio level. In this backdrop the unchallenged invocation of *replication* and/or complete *diversification* of residual risks inside a *valuation* model is dangerous and misleading – especially for new or exotic options where historical observations of option behavior are lacking. For a sell Cliquet protection trade we address risk capital using the residual risk of the attempted replication strategy found within OHMC. **Appendix-II** provides examples for sell vanilla option positions.

The expected P&L from a derivative trade should be compared with tail losses to ensure solvency, and profitability. While great trades may come from market insights that are not modeled routinely, it should be possible to weed out poor derivative trades quantitatively. To do so we define a specific solvency target and assess the risk capital associated with the derivative trade. To compute risk capital over different time intervals (derivative tenor, hedging interval, etc), we employ a *target hazard rate* to consistently assess the target survival probability over different time horizons.

survival probability over period τ	$p_s(\tau)$
τ interval hazard rate	$\lambda_\tau = \dfrac{-\ln\left(p_s(\tau)\right)}{\tau}$
τ interval hazard rate based h interval survival probability	$p_s(h) = \exp[-\lambda_\tau h]$
h interval expected P&L	$\overline{\Delta W_t(t,t+h)}$

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h interval $p_s(h)$ quintile wealth change

$$q(t;p_s(h)) \ni Probability\{\Delta W_t(t,t+h) < q(t;p_s(h))\} = p_s(h)$$

h interval $p_s(h)$ quintile deviation from average wealth change

$$Q(t;p_s(h)) = \overline{\Delta W_t(t,t+h)} - q(t;p_s(h))$$

h interval $p_s(h)$ quintile expected return on risk capital $\quad \dfrac{\overline{\Delta W_t(t,t+h)}}{Q(t;p_s(h))}$

Delta-One Bad Deal Bounds on Derivative Seller-Hedger

Our *attempted* replication via OHMC analysis imposes a zero mean change in wealth constraint. In the face of residual risks, due to inherently imperfect hedging, (which is the driver of risk capital) an option seller will need to add an additional charge over the average hedging cost to create a positive expected change in wealth and try to obtain a certain pre-expense *expected* return on risk capital. That expected return of risk capital can express an absolute solvency-profitability criteria – i.e., this trade's loss at confidence level *x* should not exceed *y* multiplied by its expected P&L. Alternatively, the solvency-profitability criteria for a derivative trade can be formulated relative to another trade – for instance simply being long the asset underlying the derivative. For the Cliquet (and other vanilla derivative trades analyzed via OHMC in the **Appendix-II**) we add a risk premium to the average cost of hedging so that they have an expected return on risk-capital equal to that of a *delta-one* long only position in the underlying.

The wealth change of the *delta-one* trader between *t* and *t*+*h* is determined by the change in asset values, the funding rates, and the discount rates:

$$\Delta W_t(t,t+h)\Big|_{delta-one-long-trader} = \left[s(t+h) - \frac{s(t)}{DF(t,t+h)}\right] df(t,t+h)$$

We assess the expected return on risk-capital of the delta-one trader and assess the bounding sell price of the derivative contract as one which results in identical expected return on risk capital for the derivative and the delta-one trader:

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$$\left.\frac{\overline{\Delta W_t(t,t+h)}}{Q(t;p_s(h))}\right|_{derivative-trader}=\left.\frac{\overline{\Delta W_t(t,t+h)}}{Q(t;p_s(h))}\right|_{delta-one-long-trader}$$

The resultant price of the derivative can be described as *the delta-one bad deal bound* pricing. At any lower price, the *delta-one* long only trader has a higher expected return per unit risk capital than the derivatives trader.

Different market operators can have different solvency targets. For the sake of illustration in this paper we use the specific choice of 1 year 99.9% confidence level. Such levels of confidence and even higher (say 1 year 99.97%) are pertinent to regulated financial institutions that rely on a perception of solvency at a high degree of confidence to maintain investor confidence and associated competitive funding costs. There can be derivative trades that look attractive (relative to delta-one long only trade) at extreme confidence levels – but less attractive at lower confidence levels, and vice-versa. From a relative value perspective one can try to price the derivative such that over a range of confidence levels it is more profitable than a delta one long position gamble on the underlying.

We examine the bounding calculation for the change in wealth over the tenor of the derivative, and over individual hedging intervals. These are two different risk-preference expressions. In employing the first one the market agent may have the capacity to take longer terms risks – possibly due to locked in funding terms and a capital base. In the latter, the market agent wants to keep score over shorter time horizons. In the case where we look at the wealth change over the derivative tenor the addition of the risk premium is straightforward - it is derived from the appropriate quintile of the total wealth change which is the sum of the wealth change over the different hedging intervals:

$$C_{bound1}=C_0(s_0)-q_{OHMC}(0;p_s(T))\times\left[\frac{\overline{\Delta W_0^{long}(0,T)}}{Q^{long}(0;p_s(T))}\right] \tag{8}$$

When we focus on the individual hedge intervals we piece together all the temporally local risk premiums rendering the trade at least as profitable as a delta-one trader locally. We discount these local premiums to trade inception, and report it as day one price difference, be it in the form of upfront or a running premium:

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$$C_{bound\,2} = C_0(s_0) - \sum_{k=0}^{K-2} q_{OHMC}\left(t_k; p_s\left(t_{k+1}-t_k\right)\right) \times \left[\frac{\overline{\Delta W_{t_k}^{long}(t_k, t_{k+1})}}{Q^{long}\left(t_k; p_s(t_{k+1}-t_k)\right)}\right] \times df\left(t_0, t_k\right) \tag{9}$$

We report sensitivities of average hedging costs and the sum of average hedging costs and the above described risk premiums.

Of course we can't claim to have exhausted all interesting relative value arguments to establish bounds on a derivative price! The market is made by agents with possibly different views on the underlying, and different utilities and risk preferences. We focus on providing an exposition that OHMC can be used to systematically integrate replication based pricing ideas and risk-preference or utility based ideas, without discarding the key tenets of either of these approaches – i.e., a derivative trader can try to replicate what she can and express a risk preference based on the residual risks inherent to attempted replication.

4. Reference Asset Description

The OHMC methodology for analyzing hedging a Cliquet is completely independent of the dynamics of the underlying asset. The ultimate practical application of the OHMC approach may even involve employing a proprietary model of the asset returns that combines empirically observed features as well as beliefs about the asset return nature. Such models also involve conditioning on observations of underlying and possibly other explanatory factors. A synthesis of econometric methods and attempted replication with quantification of average hedging costs and hedge error distributions is possible within the OHMC framework.

A well known model of asset returns that sidesteps the *perfect-hedge* contrivance even for vanilla options is afforded by a GARCH(1,1) description of asset returns (Bollersev [1986], Engle[1994]). The conditioning variable is *starting volatility*, and the jumpiness of returns associated with the return kurtosis thwarts the theoretically argued perfect hedge even under continuous hedging (see **Figure 1** and **Appendix-II**). We employ the GARCH(1,1) description to provide examples of the OHMC method as well as to describe the risk-return of the *delta-one* long only trader.

Under GARCH(1,1) the asset and its volatility evolve as follows:

$$\Delta s_k = s_k\left(\mu\Delta t + \sigma_k\sqrt{\Delta t}\,\varepsilon_k\right) \tag{10}$$

$$\sigma_k^2 = (1-\alpha-\beta)\sigma^2 + \sigma_{k-1}^2\left(\beta + \alpha\varepsilon_{k-1}^2\right) \tag{11}$$

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Standard Normal random-variates generated to create the return stochastic process in **(10)** are denoted by ε_k, and the volatility evolves per **(11)**.

Method of Moments Fitting to Empirical Returns

The empirical return statistics can be used to analytically specify the parameters of the GARCH(1,1) model for the evolution of the reference asset. Here are details of the unconditional moments used to infer the parameters (Carnero et al [2004]):

Mean and Variance

$$r \equiv \Delta s / s; \quad \bar{r} = \mu \Delta t \tag{12}$$

$$r' \equiv r - \bar{r}\ , \ \sigma_r^2 \equiv E[(r-\bar{r})^2] = \Delta t \sigma^2 \tag{13}$$

The empirical mean and variance of the returns provide direct inferences of μ and σ through (12) & (13). These could be purely historical or they could be based on ones views on the reference asset looking forward. In addition to the long-term volatility the GARCH(1,1) model also has the starting volatility as an input. This can be based on the volatility estimated from a smaller window of data trailing the date of analysis.

Kurtosis

$$\kappa \equiv \frac{E[r'^4]}{\sigma_r^4} = \frac{3}{\left[1 - \frac{2\alpha^2}{1-(\alpha+\beta)^2}\right]} \tag{14}$$

The empirical kurtosis is employed to express α as a function of β:

$$\alpha = \frac{\sqrt{(\kappa-3)(\kappa(3-2\beta^2)-3)} - \beta(\kappa-3)}{3(\kappa-1)} \tag{15}$$

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Volatility Clustering Time

$$\rho_{r'^2}(h) = \frac{E\left[\left(r'^2(t+h\Delta t)-\sigma_r^2\right)\left(r'^2(t)-\sigma_r^2\right)\right]}{E\left[\left(r'^2(t)-\sigma_r^2\right)^2\right]} = \begin{cases} \dfrac{\alpha\left[1-(\alpha+\beta)^2+\alpha(\alpha+\beta)\right]}{1-(\alpha+\beta)^2+\alpha^2} & , \ h=1 \\ (\alpha+\beta)\rho_2(h-1) & , \ h>1 \end{cases} \tag{16}$$

$$\Gamma(n) = \sum_{j=0}^{n} \rho_{r'^2}(j) \tag{17}$$

Finding the value of parameter β that best reproduces the empirical auto-covariance of the squared returns results in a complete inference of parameters. We chose the sum of the autocorrelations to a maximum lag **(17)** as the composite correlation target. This sum is referred to as the *volatility clustering time* as it quantifies a characteristic time-scale over which the volatility is correlated. This can also be described as a characteristic time-scale over which volatility fluctuations result in excess kurtosis being manifest in the realized return time series. We iterate over β from 0 to 1 to find the GARCH(1,1) value of the sum of the autocorrelations that is nearest to the empirical observation.

Three Stylized Assets

There are many facets to a Cliquet trading problem, which depend on tenor, strike, hedge interval, look-back period, in addition to the characteristics of the underlying. We will attempt to highlight the key role of the OHMC analysis for Cliquets by focusing on hedge performance and by comparing the hedged P&L distribution with that of a long market agent, to delineate relative value metrics. This is best accomplished through specific examples – for which we adopt 3 stylized descriptions of assets on which Cliquet contracts are written. These stylized asset descriptions are in **Table 2** and **Figure 3**. Asset 1 is representative of a single-stock. Asset 3 has a generic broad market index profile. Asset 2 has the profile of a diversified *alternative beta* product, be it basket of sample trades or a portfolio/index of hedge funds.

The characteristics of the stylized assets shown in **Figure 3** are: (1) correlation between the squared return residuals; (2) sample path simulation of volatility; (3) sample path simulation of return; (4) risk-capital for delta-one long only trade; (5) expected return on risk capital for delta-one long-only trade. The risk capital is assessed with a 1year 99.9% equivalent confidence level statistical target.

Stylized Asset 1

Daily Statistics	$\bar{r} = 0.000793651$; $\sigma_r = 0.025197632$; $\Gamma(252) = 10$; $\kappa = 20$
Fitted Parameters	$\mu = 0.20$ (1/yr); $\sigma = 0.40$ (1/yr$^{0.5}$); $\alpha = 0.197468$; $\beta = 0.755553$
Sensitivity Shown *	(a) $\kappa = 30$; $\alpha = 0.206672$; $\beta = 0.744686$
	(b) $\kappa = 3$; $\alpha = 0$; $\beta = 0$ (i.e., Geometric Brownian Motion)

Stylized Asset 2

Monthly Statistics	$\bar{r} = 0.008333$; $\sigma_r = 0.023094$; $\Gamma(24) = 3$; $\kappa = 8$
Fitted Parameters	$\mu = 0.10$ (1/yr); $\sigma = 0.08$ (1/yr$^{0.5}$); $\alpha = 0.354366$; $\beta = 0.419041$
Sensitivity Shown*	(a) $\Gamma(24) = 1.5$; $\alpha = 0.48795$; $\beta = 0$; $\kappa = 3$

Stylized Asset 3

Daily Statistics	$\bar{r} = 0.000476$; $\sigma_r = 0.010079$; $\Gamma(252) = 15$; $\kappa = 15$
Parameters	$\mu = 0.12$ (1/yr); $\sigma = 0.16$ (1/yr$^{0.5}$); $\alpha = 0.146813$; $\beta = 0.825871$
Sensitivity Shown *	(a) $\kappa = 3$; $\alpha = 0$; $\beta = 0$ (i.e., Geometric Brownian Motion)

Table 2. Stylized asset descriptions employed to illustrate Cliquet sensitivities.

*Sensitivities to GARCH(1,1) parameters are shown in the next section.

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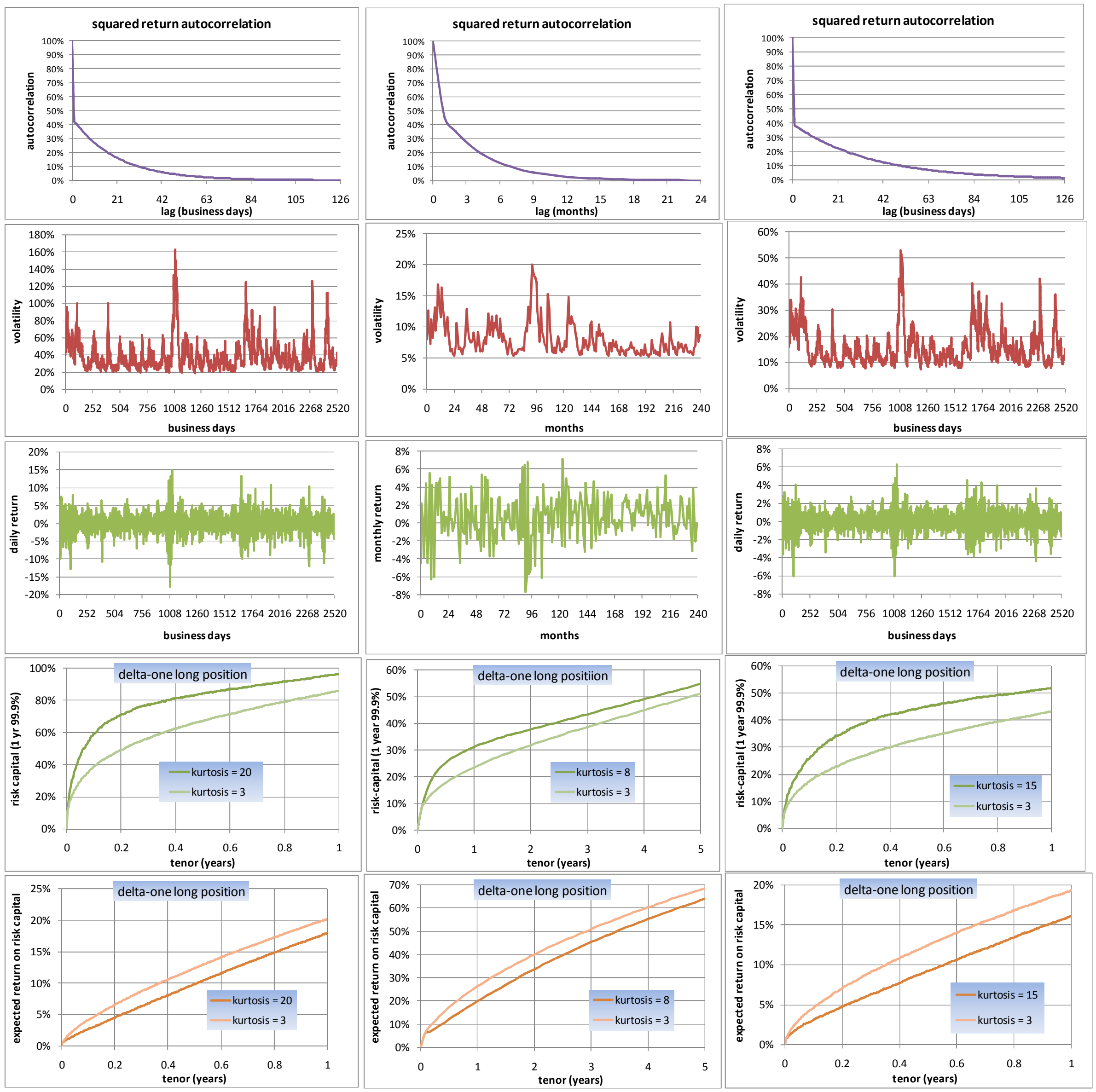

(a) Stylized Asset 1 **(b)** Stylized Asset 2 **(c)** Stylized Asset 3

Figure 3. For the three stylized assets analyzed here the auto-covariance of squared residuals, sample path simulation of the volatility and asset return, the risk capital (1 year 99.9% confidence level) and the expected change in wealth per unit risk capital for a delta-one long only trader are shown above.

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5. Expected Cost of Hedging & Residual Risks for a Cliquet Put

The multiplicity of time-scales characteristic to a Cliquet contract is generally perceived to be the main complexity over and above more vanilla contracts. It is shown in this section that in the OHMC approach developed here these additional time-scales do not create an order of magnitude additional complexity beyond the ones endemic to a vanilla contract. If one is ready to explicitly deal with fat-tails and hedging errors endemic to vanilla contracts, then the methodology of dealing with the *exotic* contract is not a world apart from the vanilla contract.

The OHMC framework is subject to being customized for a trader to express her risk preference. That risk preference – subject to revision, criticism, and *fitting* - can be readily expressed for the Cliquet contract too in OHMC. This is quite different from the presumed perfect replication *risk-neutral* paradigm where the *volatility surface* or *stochastic volatility parameters* fitted to vanilla contracts are often sought to be enforced on the exotic contract. The fitting exercise (volatility surface or stochastic volatility parameters) to vanillas in the *risk-neutral frame-work* is an unacceptable starting point to dealing with exotics because of two reasons: (1) it is based on the perfect hedge paradigm which is inconsistent with the reality of the asset behavior that gives rise to implied volatility smile-skew (2) the exotic contract can have distinct drivers of unhedgable risks compared to the vanilla contracts used to *fit* parameters.

It is surprising that the presumed perfect replication (i.e., zero-risk) model is treated as the fundamental building block to dealing with exotics in the risk neutral approach and that is widely used in accounting of P&L. The continued sponsorship of such models seems to be driven by the motivation of executing exotics that create upfront P&L when marked to market using models "calibrated" to vanillas, often with little understanding of their hedging and residual risk characteristics.

We are not offering a magical volatility surface or stochastic volatility parameters that address Cliquet trading, hedging, and pricing (see **Appendix-II** for OHMC implied volatility results). We are demonstrating a framework that requires developing an objective measure description of the asset, delineating a hedging strategy, and assessing the performance of the hedging strategy. The average cost of hedging and the hedge slippage distribution are central results of the OHMC exercise – these are needed for responsibly designing and trading exotics.

Sensitivity Analysis

The different Cliquet Contracts analyzed here are summarized in **Tables 3, 4, & 5** (**Figures 4, 5, & 6**). The starting asset value and the Cliquet payout notional are both set to $100 for the results presented here. The hedging costs and hedge slippage measures are calculated assuming upfront payments, but reported in terms of running premiums for simplicity and to facilitate comparison.

item	case:	1	2	3	4	5	6
Tenor	months	2	2	2	2	2	2
Look-back interval	weeks	1	1	1	1	1	1
Roll interval	days	1	1	1	1	1	1
Hedge interval	days	1	1	1	1	1	1
K1	%	50	75	85	95	75	75
K2	%	0	0	0	0	0	0
mu	(1/yr)	0.2	0.2	0.2	0.2	0.2	0.2
vol	(1/yr^0.5)	0.4	0.4	0.4	0.4	0.4	0.4
kurtosis	-	20	20	20	20	30	3
vol clustering time	weeks	2	2	2	2	2	2
initial hedge notional	%spot	-0.025	-0.634	-3.052	-12.688	-0.6876	-0.0002
avg duration	months	1.99	1.98	1.90	0.90	1.98	1.99
avg hedging cost	bps/yr	0.75543	28	184	2327	30	0.00493
std dev residual P&L	bps/yr	54	292	659	2211	309	2.31
bad deal bound 1	bps/yr	26	351	577	3124	361	0.077
	multiple of avg hedging cost	34.2	12.5	3.1	1.3	11.8	15.7
bad deal bound 2	bps/yr	66	478	778	3810	468	0.04676
	multiple of avg hedging cost	87.3	17.0	4.2	1.6	15.4	9.5

Table 3. Cliquet put contracts on Stylized Asset 1

Strike

The more out of money that the Cliquet contract is, the greater is the hedge slippage compared to the average hedging cost. This is shown in the strike dependence of Cliquet on Stylized Asset 1 in **Table 3** and in the **Appendix-II**. As we look at more OTM strikes, the average hedging cost decreases, the bounds on the pricing decreases, and a greater fraction of the bounding price is based on hedge slippage. This interpretation of hedging error as a part of option *value* is different from the risk-neutral practice where prices are simple averages of option payoffs under de-trended underlying descriptions. In that risk-neutral approach there is no concession made for replication errors (instantaneous hedging zero kurtosis case) and the whole distribution of the underlying is distorted to *fit* an observed price – and that distribution is labeled *risk-neutral*.

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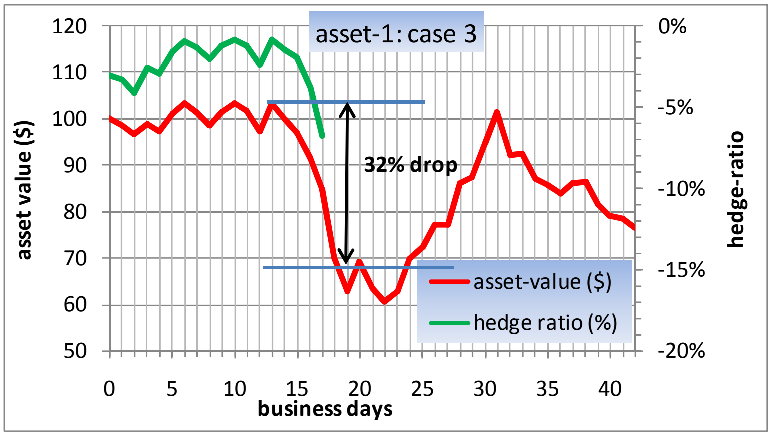
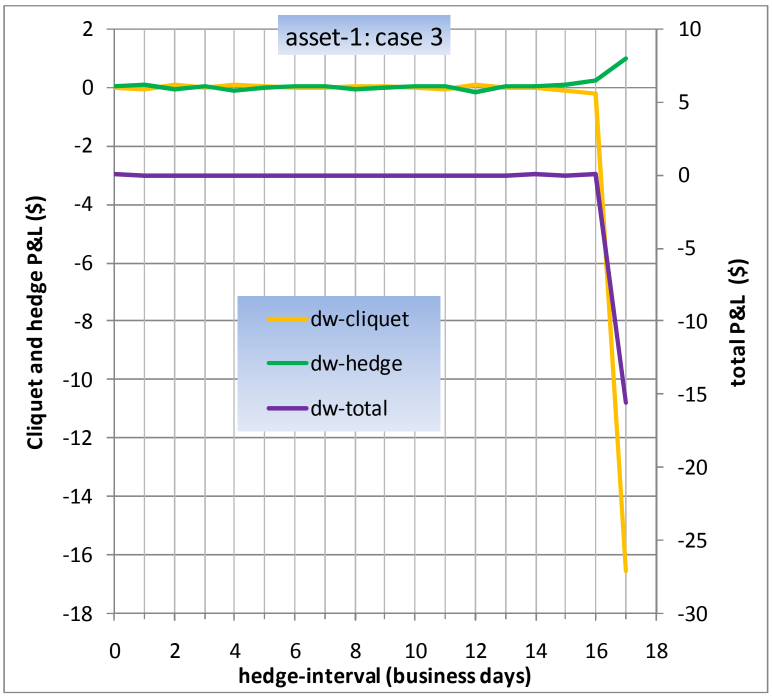
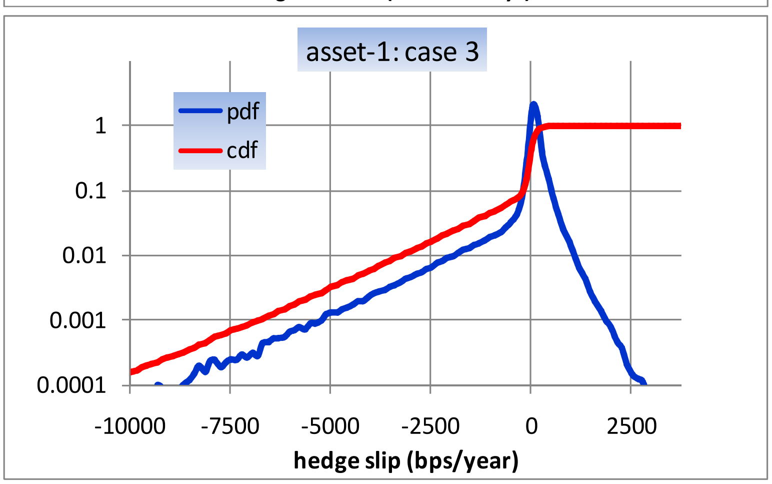


(a)



(b)



(c)

Figure 4 (a), (b), (c)

Sample path and distributional behavior for Cliquet on Stylized Asset 1 case 3

The sample path shown here corresponds to the 1 year 99.9% confidence level equivalent total wealth change over the options life – i.e., the left tail of the total change in wealth of the Cliquet put seller-hedger. The P&L plots **(b)** do not show the risk premium the option seller will add to get compensated for residual risks.

As shown in **(a)**, the payout is triggered on day 18, with the asset showing a drop of 32% over the 1 week look-back period. While the P&L variance optimal hedging amount increases from 1% to 7% over the days preceding the look-back, a loss of $15 is experienced on the day payout is triggered **(b)**. If a generally higher hedge ratio were used, then the P&L volatility over the time leading up to the trigger would be higher. While OHMC can be tailored to minimize tail losses alone, there is no perfect hedge in the face of return kurtosis which is driven from volatility fluctuations in the GARCH(1,1) model employed here.

Note that the underlying asset returns to $100 in 12 days after the Cliquet is triggered. This indicates the inherent riskiness of a Cliquet put relative to a standard European option, especially for the high kurtosis of asset 1 (kurtosis = 20, similar to that of a single stock).

The asymmetry of the total wealth change distribution **(c)** is important to recognize in deciding to sell a Cliquet put. The bounding prices (Tables 3-5) ensure that the expected wealth change of the Cliquet seller per unit risk capital (at 1yr 99.9% confidence) is no worse than a delta-1 long position (**Fig. 3**). OHMC analysis makes available all this information en-route to *valuation*.

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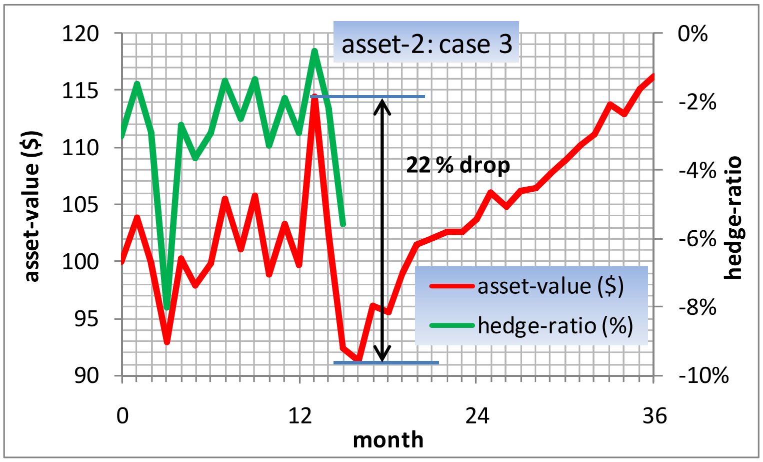
item	case:	1	2	3	4	5	6	7	8	9	10
Tenor	months	36	36	36	36	12	24	36	36	36	36
Look-back interval	months	1	2	3	4	3	3	2	3	4	3
Roll interval	months	1	1	1	1	1	1	2	3	4	1
Hedge interval	months	1	1	1	1	1	1	2	3	4	1
K1	%	85	85	85	85	85	85	85	85	85	85
K2	%	0	0	0	0	0	0	0	0	0	0
mu	(1/yr)	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
vol	(1/yr^0.5)	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
kurtosis	-	8	8	8	8	8	8	8	8	8	8
vol clustering time	months	3	3	3	3	3	3	3	3	3	1.5
initial hedge notional	%spot	-0.16	-1.25	-3.01	-3.99	-1.51	-2.48	-0.33	-0.47	-0.55	-3.74
avg duration	months	33.9	33.7	33.6	33.5	11.7	22.9	33.8	33.8	33.7	33.6
avg hedging cost	bps/yr	0.65	4.74	9.57	12.27	6.15	8.51	3.03	5.14	6.59	11.36
std dev residual P&L	bps/yr	5.97	16.64	21.21	22.18	31.92	25.47	15.48	20.83	24.54	21.95
bad deal bound 1	bps/yr	2.9	64	95	101	102	101	50	79	97	104
	multiple of avg hedging cost	4.4	13.6	9.9	8.3	16.6	11.9	16.5	15.3	14.7	9.2
bad deal bound 2	bps/yr	8.3	142	202	209	173	195	89	126	147	215
	multiple of avg hedging cost	12.6	29.9	21.1	17.0	28.0	22.9	29.5	24.5	22.4	19.0

Table 4. Cliquet contracts on Stylized Asset 2

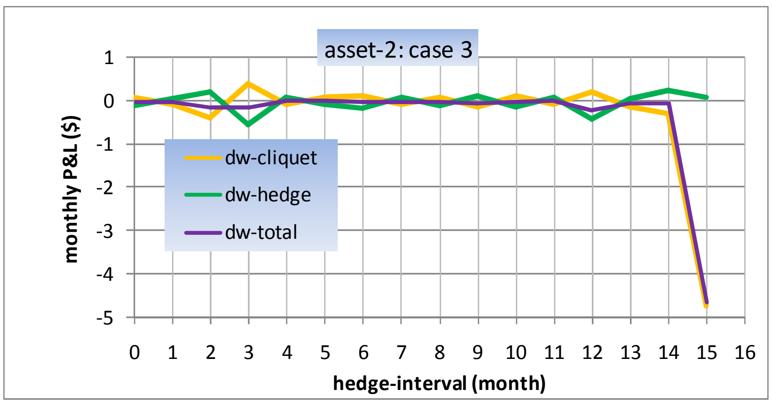
Tenor

For Cliquets we do not expect to find strong sensitivity of the running premium with tenor – as the risk of asset falling over the look-back interval is not directly influenced by the tenor. However we need to consider the volatility clustering time-scale in judging the sensitivity of the deal tenor. If the volatility clustering time-scale is a significant fraction of tenor, then the choice of starting volatility becomes important. We remind the reader that in all our examples we are setting starting volatility to the long-term historical average. So, if the volatility clustering time-scale is a significant fraction of tenor then the full range of volatility fluctuations are potentially not experienced. We witnessed that for the vanilla option in **Appendix-II**.

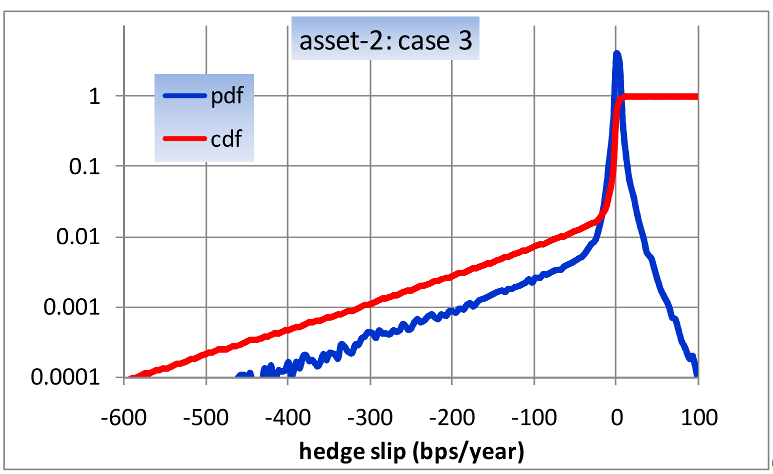
The other consideration in judging the import of tenor is the way a risk-premium is assessed. A global risk premium adds the wealth change over all the hedge intervals. The global risk premium (bound 1) is based on the sum of wealth changes over the deal tenor. The summation results in some cancellations among gains and losses and results in a tighter wealth change distribution as the averaging interval increases. In contrast, the temporally local risk premium (bound 2) assessment does not benefit from cancellations over longer tenors and therefore is expected to be less dependent on tenor, and can also become larger if the transaction tenor becomes longer than the volatility clustering time such that the full range of asset volatility and kurtosis is felt. For the Cliquet examples on stylized asset 2 we find that the global risk premium based bound decreases slightly with tenor, however the local risk premium based measure actually increases as the volatility forgets its starting value and bounces around.



(a)



(b)



(c)

Figure 5 (a), (b), (c)

Sample path and distributional behavior for Cliquet on Stylized Asset 2 case 3

As shown in **(a)**, the payout is triggered on month 16, with the asset showing a drop of 22% over the 3 month look-back period. While the P&L variance optimal hedging amount increases from 0.5% to 5.5% over the days preceding the look-back, a loss of \$4.6 is experienced on the day payout is triggered **(b)**.

Note that the nature of the Cliquet is independent of the absolute level of the asset. This asset climbed up rapidly (from \$100 to \$115 in one month) and then dropped 22 % triggering the Cliquet.

The hedge performance associated with a tail loss event (1 year 99.9% confidence level) is shown in **(b)**. Over the period leading to the trigger event the hedging works – albeit not perfectly. The P&L variance optimal hedging fails to prevent the large loss in the event of trigger in the shown sample path.

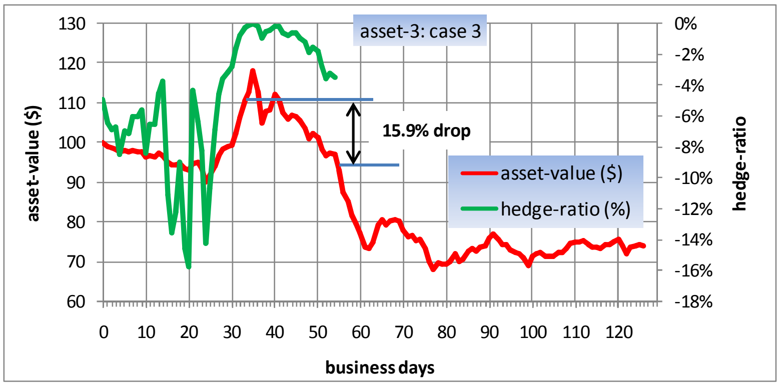
Merely focusing on tail losses may limit the losses in the event of trigger, but increase the volatility otherwise. Ways to improve the hedge performance by conditioning on additional information are briefly mentioned in the discussion section later. In no case do we expect a perfect hedge. Hence the importance of the residual P&L distribution, shown in **(c)**. At the time of trade execution, a delineation of hedging strategy and residual risks is the key result of OHMC analysis.

item	case:	1	2	3	4	5	6	7
Tenor	yr	0.5	0.5	0.5	0.5	0.5	0.5	0.5
Look-back interval	day/week/month	1 day	1 week	1 month	1 month	1 month	1 month	1 month
Roll interval	day/week/month	1 day	1 day	1 day	1 day	1 day	1 day	1 day
Hedge interval	days	1 day	1 day	1 day	2 day	3 day	4 day	5 day
K1	%	90	90	90	90	90	90	90
K2	%	80	80	80	80	80	80	80
mu	(1/yr)	0.12	0.12	0.12	0.12	0.12	0.12	0.12
vol	(1/yr^0.5)	0.16	0.16	0.16	0.16	0.16	0.16	0.16
kurtosis	-	15	15	15	15	15	15	15
vol clustering time	days	15	15	15	15	15	15	15
initial hedge notional	%spot	-0.0073	-1.04	-4.90	-4.68	-4.51	-4.32	-4.21
avg duration	years	0.495	0.485	0.454	0.454	0.454	0.454	0.454
avg hedging cost	bps/yr	0.47	16	58	58	58	58	58
std dev residual P&L	bps/yr	15	84	122	124	126	127	128
bad deal bound 1	bps/yr	26	136	174	176	179	180	183
	multiple of avg hedging cost	54.5	8.5	3.0	3.0	3.1	3.1	3.2
bad deal bound 2	bps/yr	80	350	328	297	281	271	265
	multiple of avg hedging cost	170.0	21.8	5.7	5.1	4.9	4.7	4.6

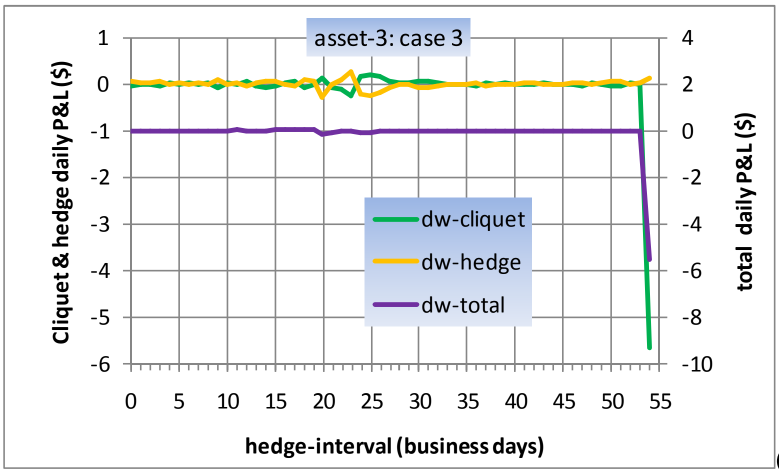
Table 5. Cliquet contracts on Stylized Asset 3

Look-Back Interval

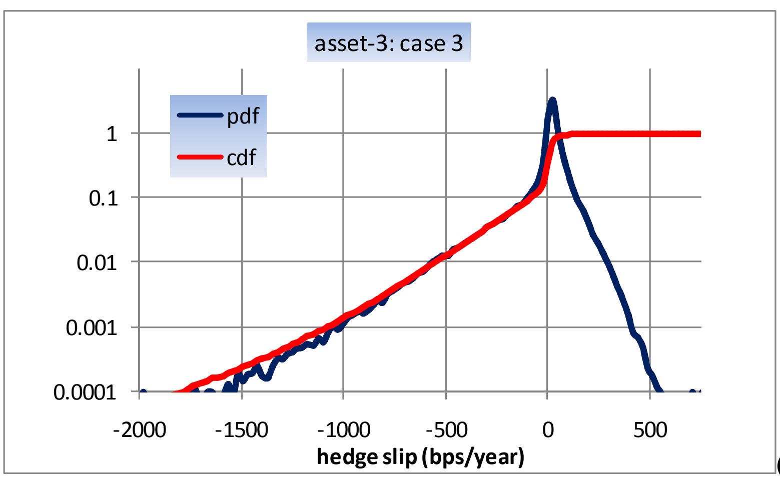
By comparing case 1, 2, and 3 of **Table 5** on asset 3 the role of look-back interval is illustrated. The average hedging cost, the hedge notional, and the residual risks increase with look-back interval. However the ratio of the residual risk to average hedging cost decreases with look-back interval. This is similar to the behavior observed for a vanilla put with respect to tenor, as shown in **Appendix-II**. This decay of the hedge slippage as a fraction of average hedging cost is likely associated with the increasing efficacy of hedging as multiple hedge adjustments are made in between the look-back interval, and in the case of bound-1, it is likely due to the effect of temporal aggregation in shrinking the width of the wealth change distribution due to cancellations of hedge slips of opposite signs.



(a)



(b)



(c)

Figure 6 (a), (b), (c)

Sample path and distributional behavior for Cliquet on Stylized Asset 3 case 3

Like shown in **Figures 4 & 5**, the sample path shown here corresponds to the 1 year 99.9% confidence level equivalent total wealth change over the options life – i.e., the left tail of the total change in wealth of the Cliquet put seller-hedger. The P&L plots do not show the risk premium the option seller will add to get compensated for residual risks.

As shown in **(a)**, the payout is triggered on day 54, with the asset showing a drop of 15.9% over the look-back period. While the P&L variance optimal hedging amount increases from 0.5% to 4% over the days preceding the look-back, a loss of \$5.5 is experienced on the day payout is triggered **(b)**. If a generally higher hedge ratio were used then P&L volatility over the time leading up to the trigger would be higher. There is no perfect hedge in the face of return kurtosis which is driven from volatility fluctuations in the GARCH(1,1) model employed here.

The asymmetry of the total wealth change distribution **(c)** is important to recognize in deciding to sell a Cliquet put. The bounding price presented in this work ensures that the expected wealth change of the Cliquet seller per unit risk capital is no worse than a simple delta-1 long position.

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Roll Interval

If the roll-interval is increased then the instances that the payout condition is checked decreases and one expects average hedging costs to go down as well as the hedge slippage measures. In **Table 4** we see that happens (despite an increasing hedging interval) by comparing case 2 and 7 and case 3 & 8. As shown in **Figure 4, 5**, a payout trigger can be associated with a rapid decline of the asset which may be followed by an increase – a larger roll interval makes the protection seller immune from some of these reversals, hence decreasing the *value* of the Cliquet put.

Kurtosis

The return kurtosis mediates the pricing dynamics in two ways: (1) higher kurtosis makes the asset values have greater probability of being realized away from the spot, hence shifting average hedging costs away from at-the-money (ATM) strikes to out-of-the-money (OTM) strikes; (2) the residual risks are controlled by kurtosis – as in the limit of continuous hedging a kurtosis value set to 3 enables the perfect hedging limit, whereas for kurtosis > 3 the residual risks can be significant. So, for some strikes that are ATM or not too OTM the pricing can become insensitive to kurtosis as it become large as the mean hedging costs shift to more OTM strikes but the hedge slippage relative to average hedging costs gets worse.

Comparing cases 2, 5, 6 in **Table 3** on the stylized asset 1 documents the impact of kurtosis going from 3 (Geometric Brownian Motion) to 20-30 which is characteristic of a typical daily return single stock time-series. For the OTM strike considered in those examples, we see a dramatic increase in average hedging cost and hedge slippage error measures in going from kurtosis of 3 to 20, and a relatively muted difference in going from 20 to 30.

Volatility Clustering Time

The volatility clustering time-scale is the time-scale over which the volatility tends to *forget* its starting value and bounces around over the range of volatilities characteristic to the GARCH(1,1) description. Note that the realized volatility of volatility is linked with realized kurtosis in the GARCH(1,1) model employed here. If the volatility clustering time is smaller than the look-back interval, then the associated return jumpiness is *seen* by the Cliquet contract and one expects higher hedging costs on the average and higher hedge slippage measures. **Table 4** presenting results on stylized asset 2 (case 2 compared with case 10) shows the impact of decreasing the volatility time-scale from the look-back interval (3 months) to half the look-back interval. We see an increase in hedging costs and hedge slippage standard deviation associated with the halving of the volatility clustering time. The tail hedge slippage measures react much more than the mean and standard deviation of hedging costs– as witnessed in the larger differences in the bounds on the Cliquet sell price. If the volatility clustering time were to become larger than the look-back interval then the fat-tails of the underlying asset (excess kurtosis) are not manifest strongly in the hedge analysis of a Cliquet.

Hedging Frequency

As shown in **Figure 1** in the introduction, for a vanilla put, the total hedging error decreases as the hedging frequency increases, albeit to a significant irreducible value for realistic asset models that exhibit excess kurtosis. We see the same feature in the Cliquet contract, as illustrated in **Table 5** case 3 through 7. In those examples the mean hedging cost is not visibly impacted, but the hedge slippage standard deviation increases with hedging interval. Bound-1 on the Cliquet price – which is based on a time-aggregated hedge slippage measure - increases with hedging interval. This is due to the widening of the loss-tail of the total change in wealth distribution with an increase in the hedging interval. However bound-2, that assesses risk premiums based on hedge error slippage over individual hedge intervals - albeit much larger than bound-1, decreases with infrequent hedging for the example shown here. The rate of decrease of bound-2 with the hedging interval decreases with an increase in the hedging interval – indicating competing influences that balance out over larger intervals. As the hedging interval increases, the 1 yr 99.9 equivalent confidence level over the hedging interval becomes less deep into the left tail of the hedge interval based wealth change distribution. So as the wealth change distribution itself is likely to be getting wider due to increasing hedging errors, by looking less deep into the left tail one can actually expect a local risk premium to be smaller. One must also remember that as the hedging interval increases, the bounding return on risk capital associated with a delta-one position also increases. These competing influences can explain the computationally simulated behavior of bound 2 as a function of the hedge interval.

6. Discussion

Relationship With Prior Work on Cliquets

Cliquet type derivatives appear in all type of exotic flavors and colors (see Gatheral [2006]): Reverse Cliquets, locally capped-globally floored Cliquets, Napoleons (a distinct French tilt in this market). Many of the early structures were fixed income in nature where the periodic coupon of the note was determined by a call or put Cliquet. A review of the market up to early 2004 has been given by Jeffery [2004]. One of the main difficulties in these derivatives is the exposure to the *forward smile-skew* of the underlying. Wilmott [2002] pointed this out and assessed that it results in a rather high sensitivity of the underlying risk model used in *pricing* such structures. As discovered by leading dealers in the late 1990's, using a local volatility model can seriously underestimate the volatility-smile sensitivity of these derivatives (a couple of dealers are rumored to have lost several 100's of millions of dollars). As mentioned by Jeffery [2004], "*dealers…failed to fully factor in hedging costs in deals written in 2002 and early 2003*". The crux of the problem was stated as follows: "*While sensitivity towards the volatility* *net change over the life of the equity option is often described as a 'second order' parameter in classical options theory* [Black Scholes] *, effectively modeling the so-called vega convexity in reverse Cliquets and Napoleon options is probably the most critical component in pricing."* In a nutshell, the sensitivity to the "volatility of volatility" is extremely high. Therefore, any model that purports to accurately *value* such structures must take into account changing volatility regimes and dynamic hedging with a view to address large sudden moves in the underlying asset. As Wilmott [2006] succinctly states, "*The way in which the volatility impacts the price of this contract* [Cliquet] *is subtle to say the least, so it makes the perfect subject for an in-depth study which I hope will reveal how important it can be to get your volatility model right"*.

After the debacle of the local volatility model in dealing with Cliquets, the knee-jerk reaction of a mainstream valuation modeling is to introduce stochastic volatility models. However, fitting parameters to expectations of payoffs with de-trended descriptions of underlying does not address replication directly – so one should not expect insights into hedging just because the underlying comes from a stochastic volatility model. To add to that, even the most popular such stochastic volatility model, the Heston Stochastic Volatility Model, have highly variable parameters while fitting to Vanilla option prices within the standard *risk-neutral* fitting framework. A comparison of stochastic versus local volatility models is given in Gatheral, [2006]. Our view is that Cliquet prices must be driven from a synthesis of *average hedging costs* plus a premium for unhedgeable risks associated with the treacheries of repeated vega-convexity flare-ups associated with the reset dates.

This work has addressed both the issues of a changing volatility and hedging analysis. The sensitivity of average hedging costs and hedge slippage to kurtosis and the volatility clustering time-scale described in the previous section squarely address points raised by Jeffrey [2004] and Wilmott [2002] & [2006]. The GARCH(1,1) parameters have been calibrated to the objective measure of the underlying. That parameter fit tends to be more stable than fitting a risk neutral model to option prices. The ability of GARCH(1,1) to represent the richness of the dataset and the return of the underlying depends on the length of the data, and one can always argue about extreme events not captured within a finite dataset, and empirical statistical characteristics not captured in GARCH(1,1). Those criticisms can be objectively adjudicated by examining datasets of different lengths and/or out-rightly specifying statistical characteristics that are desired and refining the objective measure description (including adding a death state). We are not adverse to the Heston model. In fact we use the Heston model (among others) within the OHMC framework - but calibrated to the objective measure of the underlying (using a similar method of moment matching as in the GARCH methodology in **section 4**). We do not find any value in the calibration of risk-neutral models to vanilla option prices (without analyzing hedging and residual risks) and making assessments of valuation of Cliquets or other exotics, presuming perfect replication. That does not address the risk-return dynamics of the Cliquet contract or any other exotic derivative.

OHMC Approach & Exotics

The recognition of the idea of replication and the evolution of quantitative approaches from the Bachelier price (average of future cash-flows) to a replication cost price (cost of mitigating risks) is widely reflected in current quantitative finance practice. The idea of replication is powerful and convenient. Indeed, *when perfect replication is feasible*, one can simply take a statistical *average* of the option payoff under a *de-trended* description of the underlying asset (relative to carry cost) and the user of the model does not have to deal with any statistical risk measure (other than the average of a hypothetical distribution). As the marketplace gives birth to complex derivatives at a rapid pace, *implied* parameters fitted to observed prices of vanilla options are often used in the *valuation* of complex derivatives by taking statistical averages of its payoff with a *de-trended* underlying. The upfront P&L and/or carry for an exotic derivative, found by using a *valuation* model employing parameters fitted to vanillas, are of great interest to businesses.

In this evolution of modeling of derivative contracts, limits on replication are a major *inconvenience*. If the vanilla derivative contract cannot be practically replicated, then its price should/could reflect a mix of some average hedging cost plus possibly a risk premium for unhedgable risks. In fitting a parameter of a presumed perfect replication model to the observed prices of a far from perfectly replicable option one may end up *believing* that the prevailing risk premiums are being *correctly* or *adequately* represented, and will be effectively propagated to the exotic derivative valuation. This is how valuation models create an appearance of an intelligent exercise of risk aversion. However at no stage is risk actually being assessed in this cascading use of averages of payoffs under *de-trended descriptions*. The asymmetries of residual risks and the market signals about them in the bid-offer of vanilla contracts are not explicitly propagated into the value of the exotic derivative by taking averages under de-trended descriptions. The increase in complexity of the derivative contracts is therefore not accompanied by a better understanding of the hedging strategy or a quantification of residual risks in this mode of *valuation modeling* that has taken hold of accounting practices for simple and complex derivatives. The purported unique price assessed using such valuation models and the ensuing implications for upfront P&L and/or carry beclouds the substantive business risk management issues of residual risks inherent in any attempted replication and assessment of risk-return. Thus we find ourselves surrounded by a plethora of "risk-neutral" *valuation* models whose main claims are convenience and rapidity of fitting – with no analysis of hedging, attempted replication and residual risks being done en-route to *valuation*.

Alternatively, as pursued in this paper, one can start with a description of the underlying and explicitly analyze hedging and try to find the best hedging strategy as done in the OHMC approach. If the hedging strategy is perfect, then the cost of hedging is the value of the derivative. On the other hand if the hedge performance is not perfect, then the probability

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distribution of the residual P&L is useful. The OHMC method is mindful of the power of replication, but puts the burden of invoking it on the designer of the hedging strategy and tempers it with an estimate of the hedge slippage, accounting for characteristics of the derivative contract when the underlying asset exhibits jumpy returns. While the routine water cooler talk with a savvy trader or risk manager could reveal the catalogue of unhedgable risks endemic to an attempted replication strategy, the mainstream quantitative *valuation* models have found it difficult to resist the allure of *presuming* perfect replication. This aversion to attempting to deal with the reality of *imperfect replication* is perhaps due to the steep gradient in going from taking averages under de-trended underlyings, to actually articulating a hedging strategy and elucidating its limitations. Another reason mainstream quantitative valuation modeling has avoided dealing with *imperfect replication* is, perhaps, that imperfect replication challenges the notion of an unassailable or unique model price, which has become the main accounting goal of mainstream valuation models. Rather than abandon addressing the imperfect replication situation, OHMC provides assessments of average hedging costs and residual hedging errors that can be used by a market agent in judging the price at which a market opportunity presents itself. While OHMC in itself does not address the diversifiability of residual risks, or the risk preferences of distinct market agents, the information it provides can be used by a market agent to express her risk preference in the contexts of her trading book, and in effecting a hedging strategy.

We believe that the derivatives trader and his business and risk managers benefit from the in-depth hedge performance analysis that is provided by OHMC, prior to *pricing* any derivative trade. In using OHMC, results on hedge performance and residual risks are available along with any assessment of *valuation* and the ensuing P&L (or trade carry). As a result, the upfront P&L and carry of the *hedged* derivative position can be readily compared with irreducible hedging errors. Hence a relative value metric for the trade is available as a part and parcel of *valuation* using OHMC. Thus, the OHMC framework is a suitable tool for providing a consistent view of derivatives trades to trading, risk-control, and product-control.

Future Work

For problems with one major risk factor, the model developed here demonstrates the practical feasibility of OHMC for any option problem, including path dependent problems. The key to efficient implementation is recognizing the limited support of the basis functions (**Appendix-III**) while assembling the set of equations to effect the numerical solution to the variational problem (**Appendix-I**). Employing total wealth change quintiles as calibration targets, rapid fitting (by post-processing OHMC algorithm output) to observables can also be achieved – but the main benefit being the availability of hedge slippage measures at the time of *pricing*. So the OHMC based *valuation* model is a tool of trading strategy and risk management while it is being used to mark-to-market a trading book. Extensions to other hedging error measures – say *expected shortfall* – are also feasible.

Based on this work, we also believe that OHMC can be applied to practical applications involving multidimensional problems, including multi-name credit derivatives. For example, results on static hedge optimization for synthetic CDOs have been reported by Petrelli et al, [2006] and implications of uncertainty of *realized correlation* of spreads on hedging trades that are long correlation (and long carry) are assessed in Petrelli et al [2007]. Any sort of 'correlation-trading' (best-of-baskets, multi-asset options, CDOs) should benefit from understanding the role of uncertainty in realized correlation on hedge slippage, in addition to the hedging error driven by marginal fat tail distributions. Providing a practical primer on hedging while 'correlation-trading' and the ensuing implications for risk-return should be a fruitful target of multidimensional applications for OHMC.

Even in seemingly single asset problems, such as the Cliquet analysis pursued here, conditioning on explanatory variables could make the hedging strategy more effective. For instance, if there is temporal persistence of volatility, then treating the squared residuals of return as a second variable (not necessarily traded) that is observed and accounted for in the hedge optimization holds the prospect of improving hedge performance. Such conditioning can be effected in the multidimensional OHMC framework. We will next report on efficient two and three dimensional OHMC implementations and present examples employing multi-dimensional jumpy assets.

Acknowledgements. The authors gratefully acknowledge Joseph Hedberg for reviewing drafts of this document, and Ram Balachandran for an efficient implementation of the OHMC algorithm described in Appendix I & III.

JEL Classification

G13, G11, D81

G13	Contingent pricing; Futures Pricing
G11	Portfolio Choice; Investment Decisions
D81	Criteria for Decision-Making under Risk and Uncertainty

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Appendix-I

Cliquet Hedging Variational Problem

The change in wealth of a Cliquet protection seller-hedger follows

$$\Delta W_{t_k}(t_k, t_{k+1}) = C_k(s_k) - G_k + \Phi_k(s_k) H_k \tag{I-1}$$

$$G_k = (1 - I(t_k, t_{k+1}])C_{k+1}(s_{k+1}) df(t_k, t_{k+1}) + \omega(t_k, t_{k+1}] - \eta \Psi \chi(t_k, t_{k+1}) \tag{I-2}$$

$$H_k = (s_{k+1} - s_k / DF(t_k, t_{k+1})) df(t_k, t_{k+1})$$

In (I-1) & (I-2) $C_k(.)$ and $\Phi_k(.)$ are unknown functions that we seek to find to accomplish the following statistical goal:

$$E[\Delta W_{t_k}(t_k, t_{k+1})] = 0 \tag{I-3a}$$

Minimize: $$\sigma^2_{\Delta W_{t_k}(t_k, t_{k+1})} = E[\{\Delta W_{t_k}(t_k, t_{k+1}) - \overline{\Delta W_{t_k}}(t_k, t_{k+1})\}^2] \tag{I-3b}$$

Minimizing the wealth change variance with a zero mean change of wealth is equivalent minimizing the mean squared change in wealth with a zero mean change of wealth. To render this variational problem finite-dimensional we represent the option value and hedge notional functions of spot in a finite dimensional representation

$$C_k(s_k) = \sum_{j=0}^{M_C - 1} a_j^k A_j(s_k) \qquad \Phi_k(s_k) = \sum_{j=0}^{M_\Phi - 1} b_j^k B_j(s_k) \tag{I-4}$$

The finite-dimensional representation used for the computations of the main section is detailed in **Appendix-III**. Using (I-1) through (I-4) the first two statistical moments of the option-seller-hedger's wealth change follow

$$E[\Delta W_{t_k}(t_k, t_{k+1})] = \sum_{j=0}^{M_C - 1} a_j^k E[A_j(s_k)] - E[G_k] + \sum_{j=0}^{M_\Phi - 1} b_j^k E[B_j(s_k) H_k] \tag{I-5}$$

$$E[\Delta W_{t_k}(t_k, t_{k+1})^2] = E\left[\left\{\sum_{j=0}^{M_C - 1} a_j^k A_j(s_k) - G_k + \sum_{j=0}^{M_\Phi - 1} b_j^k B_j(s_k) H_k\right\}^2\right] \tag{I-6}$$

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To solve for the unknown coefficients a_j^k and b_j^k we employ a Lagrange multiplier technique:

$$F_k = E[\Delta W_{t_k}(t_k, t_{k+1})^2] + 2\gamma E[\Delta W_{t_k}(t_k, t_{k+1})] \tag{I-7}$$

$$\frac{dF_k}{da_i^k} = 0 \qquad 0 \le i \le M_C - 1 \tag{I-8}$$

$$\frac{dF_k}{db_i^k} = 0 \qquad 0 \le i \le M_\Phi - 1 \tag{I-9}$$

$$\frac{dF_k}{d\gamma} = 0 \tag{I-10}$$

Substituting (I-5) and (I-6) into (I-7)-(I-10) defines the set of linear equations that must be solved to solve the finite-dimensional approximation of the Cliquet optimal hedge-valuation variational problem. Further details of these linear equations are provided here. (I-8) through (I-10) can be expressed as

$$G_{ij} h_j = q_i \quad 0 \le i, j \le M_C + M_\Phi \tag{I-11}$$

A sum over the repeated index j is implied in the above (I-11). The vector of unknowns is represented by h_j:

$$\begin{aligned} h_j &= a_j^k && 0 \le j \le M_C - 1 \\ h_j &= b_{j-M_C}^k && M_C \le j \le M_C + M_\Phi - 1 \\ h_{M_C + M_\Phi} &= \gamma \end{aligned} \tag{I-12}$$

From (I-8)

$$\begin{aligned} &\text{for } 0 \le i \le M_C - 1 \\ &q_i = E[A_i(s_k) G_k]; \\ &G_{ij} = E[A_i(s_k) A_j(s_k)] && 0 \le j \le M_C - 1 \\ &G_{ij} = E[A_i(s_k) B_{j-M_C}(s_k) H_k] && M_C \le j \le M_C + M_\Phi - 1 \\ &G_{ij} = E[A_i(s_k)] && j = M_C + M_\Phi \end{aligned} \tag{I-13}$$

From (I-9)

$$\begin{aligned} &\text{for } M_C \le i \le M_C + M_\Phi - 1 \\ &q_i = E[B_{i-M_c}(s_k) G_k H_k] \\ &G_{ij} = E[B_{i-M_C}(s_k) A_j(s_k) H_k] && 0 \le j \le M_C - 1 \\ &G_{ij} = E[B_{i-M_C}(s_k) B_{j-M_c}(s_k) H_k{}^2] && M_C \le j \le M_C + M_\Phi - 1 \\ &G_{ij} = E[B_{i-M_c}(s_k) H_k] && j = M_C + M_\Phi \end{aligned} \tag{I-14}$$

From (I-10)

$$
\begin{aligned}
&\text{for } i = M_C + M_\Phi \\
&q_i = E[G_k] \\
&G_{ij} = E[A_j(s_k)] && 0 \le j \le M_C - 1 \\
&G_{ij} = E[B_{j-M_c}(s_k)H_k] && M_C \le j \le M_C + M_\Phi - 1 \qquad \text{(I-15)} \\
&G_{ij} = 0 && j = M_C + M_\Phi
\end{aligned}
$$

Initial Time Step

Solving for the pricing and hedge notional is an ordinary minimization problem at the first time step:

$$\Delta W_{t_0}(t_0, t_1) = C_0 - G_0 + \Phi_0 H_0 \tag{I-16}$$

C_0 and Φ_0 are the unknown quantities in (I-16). We define perturbed quantities as deviations around ensemble averages

$$G_0^{'} = G_0 - \overline{G_0}; H_0^{'} = H_0 - \overline{H_0}$$

The solution for C_0 and Φ_0 that enforce zero mean change in wealth and that minimize the wealth change variance are

$$\Phi_0 = \frac{\overline{G_0^{'} H_0^{'}}}{\overline{H_0^{'2}}};\; C_0 = \overline{G_0} - \Phi_0 \overline{H_0} \tag{I-17}$$

The OHMC algorithm to solve the variational problem for a Cliquet is not too different from that to solve the vanilla equity option hedging problem. We start from the option maturity and work backwards, solving for the option spread value and optimal hedge notional. Payout trigger events and the hedging interval are explicitly accounted for in the wealth balance. For the Cliquet hedging problem, all the statistical averages are conditioned on knockout not having occurred at the starting time step. The optimal hedge ratio and average hedging cost value are conditioned on there being a live Cliquet contract.

Consistency Between Vanilla and Exotic Options

The similarity of OHMC algorithm between vanilla options and exotics is attractive because it enables *consistency* in the analysis of vanilla options and more exotic options. This consistency is further reinforced if one uses identical descriptions of the process underlying the derivative contract. If the exotic option is sensitive to a particular time scale description of the process, or, extreme tail behavior, then one needs to refine the description of the underlying based on available empirical information or ones view. That view can be a proprietary view and developed by comparison with other assets for which more empirical information is available. The vanilla option analysis may also benefit (or at least not hurt) from such a refined stochastic description of the underlying. The hedge ratios resulting from the OHMC analysis have a concrete real-world objective– and there is no need to *switch measures* and descriptions of underlying between *valuation* analysis and assessment of hedge ratios. The model parameters fitted to vanilla prices (without addressing irreducible hedging errors) do not have to be imposed on the exotic derivative under the superficial guise of consistency.

It is our experience that exotics can have distinct sensitivities that control the hedging errors in attempting to replicate them, and the option trader needs to directly focus on them, rather than being bound by say a volatility surface that originates from fitting volatilities to vanillas. This is practically important because often upfront P&L resulting from the imposition of volatility surfaces that fit vanillas onto exotics can end up becoming the motivation for doing a trade, rather than an argument based on risk-return of the exotic option. In the OHMC framework, consistency of analysis of vanilla and exotic option requires the following:

1. Employ the same empirically realistic description of the underlying to the vanilla and exotic derivative problem;
2. Assess the optimal hedging strategy for both the vanilla and exotic along with the average hedging costs;
3. Assess residual hedging errors – be they driven by fat tails, or by discrete hedging intervals for both the vanilla and the exotic option;
4. Where market prices for the vanilla option are readily available, interpret them based on risk return metrics derived from average hedging costs and residual risks, and develop a picture of market risk aversion that is cognizant of demand and supply for the derivative;
5. Develop a view of exotics *pricing* based on average hedging costs and deviations around that average. If there are common elements of risk-return between an exotic and a vanilla derivative, then the vanilla derivative observable pricing can help guide the exotic option trader to a competitive pricing point and the associated risk-return of the trade.

Appendix-II

Optimal Hedge Analysis of Vanilla Options

Examples of variance optimal hedging analysis of sell vanilla option positions are presented here. The stylized asset descriptions employed in the main text (**Section 4**) are used here. We present hedging costs and residual risks arising from a combination of asset return kurtosis and discrete hedging interval. We also compare the P&L distribution of a trader that sells options and hedges to that of a *delta-one* trader that is simply long the underlying asset. Those relative value metrics are also cast in the form of bonds on the option price (detailed in the main section) and corresponding bounding *implied volatility surfaces*.

In applying *risk-neutral* models to value exotics, often the starting point is *fitting* their parameters to the observed prices of vanilla calls and puts. Unlike the risk-neutral formalism, OHMC method does not presume perfect replication and makes available to the user the residual risks in attempted replication. OHMC enables interpreting the observed prices as a combination of average hedging costs and the distribution around that average. From a pure accounting *mark-to-market* perspective if the objective is merely to fit a specific price, then one can calibrate to the different quintiles around the average hedging cost in OHMC. The convenience of *fitting* the risk neutral formalism based models to derivative price data should not be an excuse for not knowing hedging errors endemic to attempted option replication. OHMC offers an ability to fit prices too, although that is not its main goals. More importantly, OHMC develops and back-tests the hedging strategy and makes available average hedging costs and hedging errors.

OHMC Methodology

Denoting the optional value and hedge ratio at time t_k by C_k and Φ_k, the change in wealth of the option seller-hedger follows:

$$\Delta W_{t_k}(t_k, t_{k+1}) = \Delta W_{t_k}^{option}(t_k, t_{k+1}) + \Delta W_{t_k}^{hedge}(t_k, t_{k+1}) =$$

$$C_k(s_k) - C_{k+1}(s_{k+1}) df(t_k, t_{k+1}) + \Phi_k(s_k)(s_{k+1} - s_k / DF(t_k, t_{k+1})) df(t_k, t_{k+1}) \qquad \text{(II-1)}$$

Using the symbols that are convenient in the Cliquet OHMC algorithm, we rewrite (II-1) as

$$\Delta W_{t_k}(t_k, t_{k+1}) = C_k(s_k) - G_k + \Phi_k(s_k) H_k$$

$$G_k = C_{k+1}(s_{k+1}) df(t_k, t_{k+1}); \; H_k = (s_{k+1} - s_k / DF(t_k, t_{k+1})) df(t_k, t_{k+1}) \qquad \text{(II-2)}$$

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The OHMC solution is propagated from maturity to the option starting point, where it becomes an ordinary minimization problem, as formulated in **Appendix-I**.

I. Hedging Error Dependence on Hedging Interval & Return Kurtosis

Example 1. Sell Put and Delta Hedge Stylized Asset 3

Underlying characteristics: $\mu = 0.12$ (1/yr); $\sigma = 0.16$ (1/yr$^{0.5}$); $\Gamma(252) = 15$; $\kappa = 15$

Put parameters: spot = 100; strike = \$100; r = 4%/yr; maturity = 1 month

The disparity in hedge slippage error between a realistically fat-tailed distribution and an idealized return distribution with no kurtosis is visible in **Figure 1 (main section 1)**. Kurtosis > 3 renders the hedging error irreducible as the hedging frequency increases. For Geometric Brownian Motion the hedging error decreases much more rapidly with hedging frequency, and is headed to zero in the limit of continuous hedging. This result shows how much of the mathematical machinery to deal with continuous hedging while ignoring realistic return fat tails is of marginal practical importance because in the limit of continuous hedging the transaction costs would become unboundedly large, and in practically realistic returns, after a point, hedging more often does not reduce the hedging errors. The OHMC framework handles discrete hedging without having to assume zero excess kurtosis for the underlying.

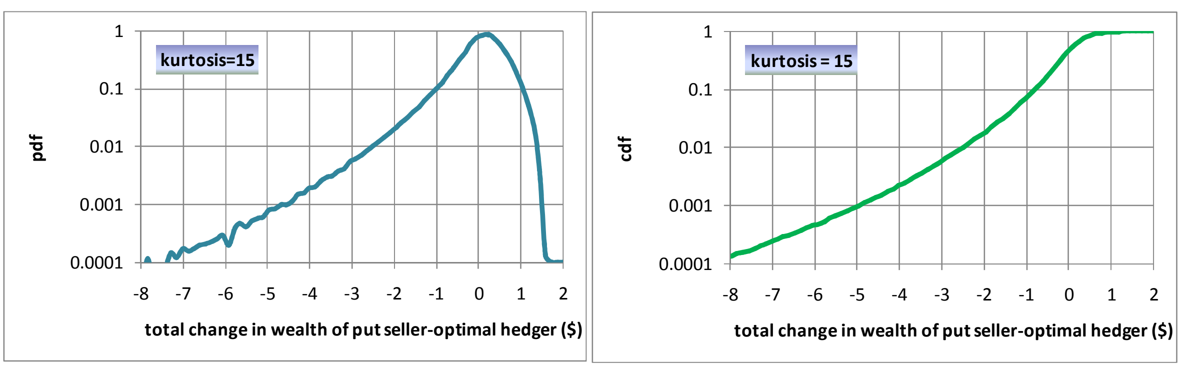

Figure II-1. Probability and cumulative density functions of the total wealth change $\Delta W_0(0,T)$ of the put seller-daily variance optimal hedger. The OHMC algorithm imposes a zero mean change in wealth $E[\Delta W_0(0,T)] = 0$ and minimizes $E[\Delta W_{t_k}(t_k, t_{k+1})^2] = 0$ over every hedge interval. The average cost of hedging, C_0, and the residual P&L distribution is provided by OHMC by "back-testing" the trading strategy as a part of the MC simulation. In this work we report multiple measures of value: (1) average hedging cost; (2) average hedging cost plus risk premium that renders the expected return on risk capital identical to a trade that is long the underlying asset. The risk premium is assessed over the whole deal life (giving rise to bound 1), and alternatively, over each hedging interval (bound 2), as detailed in the main section (see **section 3** for a full description of these bounds).

II. Strike-Kurtosis Sensitivity

<u>Example 2. Sell Put and Daily Delta Hedge on Stylized Asset 1</u>

Underlying characteristics: $\mu = 0.20$ (1/yr); $\sigma = 0.40$ (1/yr$^{0.5}$); $\Gamma(252) = 10$; $\kappa = 20$

Put parameters: spot = \$100; strike = K; r = 4%/yr; maturity = 1 month

strike K (\$)	hedge notional (% spot)	average hedging cost		delta-one risk-return based bad deal bound for option-seller					
				1 year 99.9% equivalent quintile of sum of hedging errors based bound (1)			Sum of 1 year 99.9% equivalent quintile of hedging errors based bound (2)		
		% spot	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)
80	-4.6	0.185	44.7%	0.607	3.29	58.0%	0.513	2.78	55.6%
90	-17.5	1.002	39.8%	1.580	1.58	47.5%	1.440	1.44	45.7%
100	-46.4	4.176	37.8%	4.858	1.16	43.7%	4.685	1.12	42.2%
110	-75.4	10.987	38.8%	11.706	1.07	46.7%	11.513	1.05	44.6%
120	-90.2	19.997	41.7%	20.733	1.04	55.1%	20.501	1.03	51.5%

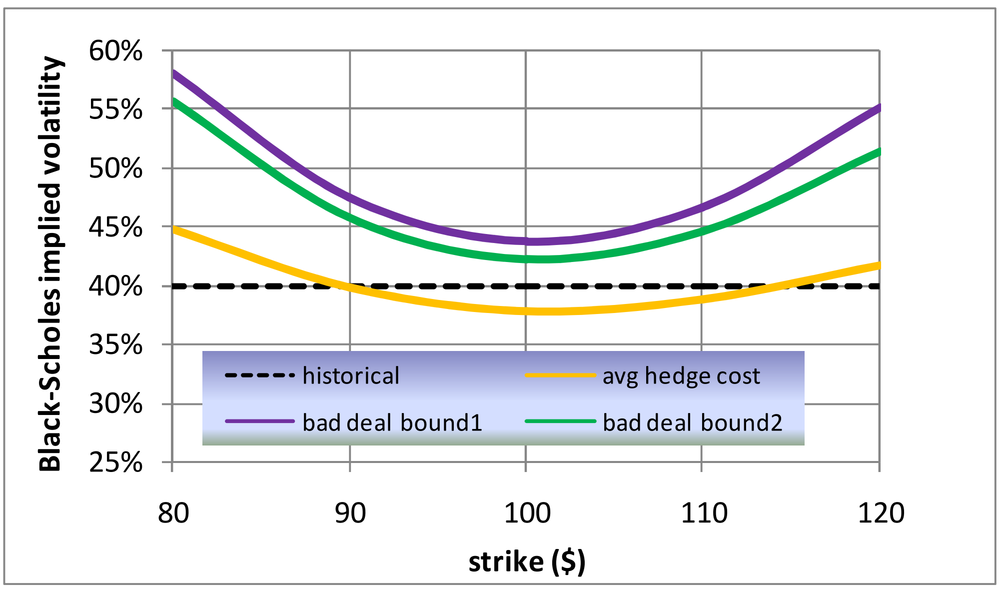

Table & Figure II-2. OHMC results for a sell 1 month put with daily delta hedging (Example 2). OHMC evaluates the P&L variance optimal hedge ratio and the residual hedge errors. The *bad deal bound* for the seller is the put sell price below which the put sellers expected return on risk capital (at 1 yr 99.9% confidence level) is below that of a simple long position in the asset. The risk capital is assessed based on the hedging errors that are residual in the variance optimal hedging strategy. The bad-deal bound-1 is based on equating the option seller-hedgers return on risk capital to a delta-1 trade over its life – whereas bad-deal bound-2 simply focuses on the discrete hedging interval risk-return.

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Kurtosis Sensitivity

The central role of return kurtosis in thwarting perfect replication and creating a dependence of implied volatility with strike is demonstrated in the results presented in this Appendix (**Figure 1** main section & **Figure II-1** and **II-2**). Now we examine the impact of changing the kurtosis, keeping all else equal. Below are shown results for Example 2 with $\kappa = 30$. The increase in kurtosis results in a greater chance that the asset values will end up further away from its average value – hence the average hedging around the spot decreases, but increases away from spot. However as shown in the first example, the residual risks are also an increasing function of kurtosis. The option seller's bounds on pricing, found by adding residual risks to the average hedging cost, increased for all the strikes for Example 2.

strike K ($)	hedge notional (% spot)	average hedging cost		delta-one risk-return based bad deal bound for option-seller					
				1 year 99.9% equivalent quintile of sum of hedging errors based bound (1)			Sum of 1 year 99.9% equivalent quintile of hedging errors based bound (2)		
		% spot	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)
80	-4.70	0.188	44.9%	0.625	3.33	58.4%	0.523	2.78	55.9%
90	-17.58	1.000	39.8%	1.596	1.60	47.7%	1.442	1.44	45.7%
100	-46.39	4.164	37.7%	4.866	1.17	43.8%	4.681	1.12	42.2%
110	-75.38	10.981	38.7%	11.715	1.07	46.8%	11.511	1.05	44.6%
120	-90.06	19.999	41.7%	20.754	1.04	55.4%	20.506	1.03	51.5%

Table II-3 OHMC results for a sell 1 month put with daily delta hedging (Example 2 with $\kappa = 30$).

In this example the bad deal bound-1 is larger than bad deal bound-2. In comparing bound 1 and bound 2, two factors need to be kept in mind:

- In bound 1 the residual hedging errors are summed up over the option tenor and the total hedging error tail is employed to find the derivative sell price that makes the expected return on risk capital over the option tenor identical to a delta-one long position

- In bound 2 the residual hedging errors over each hedging interval is analyzed separately to assess an addition to the average hedging cost that ensures the return on risk capital over every hedge interval is equal to a delta-one long position over the corresponding hedge-interval

In the next example the pricing bad deal bound 1 falls below the bad deal bound 2.

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III. Tenor & Volatility Clustering Time Sensitivity

Example 3. Sell Put & Monthly Delta Hedge Stylized Asset 2

Underlying characteristics: $\mu = 0.10$ (1/yr); $\sigma = 0.08$ (1/yr$^{0.5}$); $\Gamma(24) = 3$; $\kappa = 8$

Put parameters: spot = \$100; strike = 85; r = 4%/yr; maturity = T

T (months)	hedge notional (% spot)	average hedging cost		delta-one risk-return based bad deal bound for option-seller					
				1 year 99.9% equivalent quintile of sum of hedging errors based (bound 1)			Sum of 1 year 99.9% equivalent quintile of hedging errors based (bound 2)		
		% spot	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)
3	-0.15	0.00223	11.60	0.09782	43.91	18.35	0.16491	74.03	20.18
6	-1.17	0.02366	11.06	0.42587	18.00	18.33	0.52214	22.07	19.30
9	-2.19	0.05499	10.62	0.68339	12.43	17.62	0.80604	14.66	18.46
12	-3.07	0.09006	10.37	0.90596	10.06	17.17	1.05451	11.71	17.98
18	-4.21	0.15499	10.06	1.20435	7.77	16.38	1.36099	8.78	17.03
24	-4.86	0.20996	9.91	1.32973	6.33	15.57	1.62705	7.75	16.63

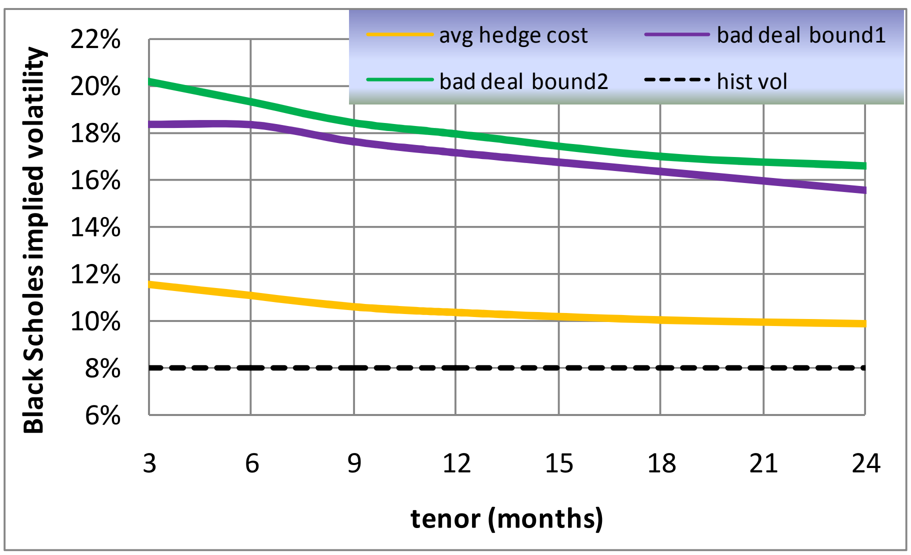

Table & Figure II-4. OHMC results for a sell 85% strike put with monthly delta hedging (Example 3). OHMC evaluates the P&L variance optimal hedge ratio and the residual hedge errors. The *bad deal bounds* for the seller is the put sell price below which the put sellers expected return on risk capital (at 1 yr 99.9% equivalent confidence level) is below that of a simple long position in the asset. The risk capital is assessed based on the hedging errors that are residual in the variance optimal hedging strategy. The first bad deal bound assesses risk-capital based on the sum of LIBOR discounted hedging errors over the deal life. The second bad deal bound is based on assessing risk capital over the hedging interval and adding them to the upfront price (discounted at LIBOR).

Volatility Clustering Time Sensitivity

To illustrate the impact of the volatility clustering time-scale we show results with a smaller volatility clustering time. We keep all the parameters identical, but set the volatility clustering time to one-and-a-half months: $\Gamma(24)=1.5$ instead of 3 months in the base case. The starting volatility is the same as the previous case- i.e., set to the long-term volatility. The effect of lowering the volatility clustering time is that the volatility bounces around over shorter time intervals. As a result, the hedging errors are expected to become larger over shorter tenors as the starting volatility is forgotten and the full gamut of volatilities are realized. Due to the shorter memory of the volatility clusters we also anticipate that the temporal averaging of volatility will occur more effectively over a given tenor compared to the case of more persistent volatility.

T (months)	hedge notional (% spot)	average hedging cost		delta-one risk-return based bad deal bound for option-seller					
				1 year 99.9% equivalent quintile of sum of hedging errors based (bound 1)			Sum of 1 year 99.9% equivalent quintile of hedging errors based (bound 2)		
		% spot	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)	% spot	multiple of avg hedge cost	Black-Scholes implied vol (%)
3	-0.27	0.00412	12.27	0.167	40.53	20.23	0.221	53.54	21.39
6	-1.29	0.02573	11.18	0.503	19.57	19.12	0.565	21.95	19.69
9	-2.05	0.04944	10.46	0.695	14.06	17.71	0.815	16.48	18.52
12	-2.70	0.07653	10.10	0.846	11.06	16.83	1.016	13.27	17.77
18	-3.48	0.12522	9.69	1.007	8.04	15.50	1.266	10.11	16.63
24	-3.86	0.16498	9.47	1.053	6.38	14.50	1.492	9.04	16.16

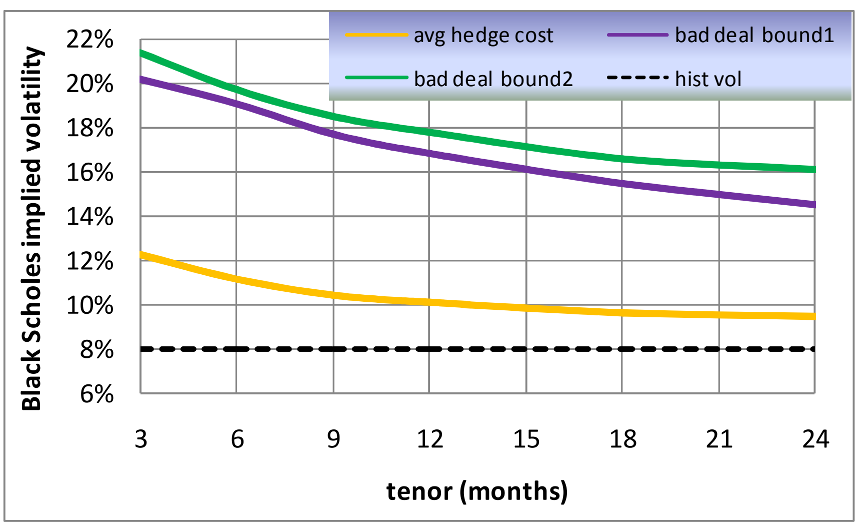

Table & Figure II-5. OHMC results for a sell 85% strike put with monthly delta hedging (Example 3). The parameters are identical to the previous example – other than the volatility clustering time is shorter (1.5 months instead of 3 months). As a result, over smaller tenors the impact of volatility fluctuations is felt, yielding a higher value of the put. The temporal aggregation of the volatility also occurs more rapidly, giving rise to a sharper decay of implied volatility versus tenor.

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Appendix-III

Piecewise Hermite Cubic Basis Functions

The hedge optimization problem posed here is infinite-dimensional insofar as we seek to find how the hedge ratio and option value should depend on spot, so that the expected change in wealth is zero and the hedge error measure is as small as possible. This calculus of variations problem is rendered numerically tractable by using finite dimensional representations of the hedge ratio and option value.

Consider a finite dimensional representation of a function $f(x)$. The x space is discretized by an increasing sequence of M nodal values, x_j, $0 \le j \le M-1$, and basis functions are chosen to have limited support around these nodal locations and provide the desired level of continuity at the nodal locations. Piecewise cubic Hermite polynomials ensure continuity up to the first derivative at the nodal locations. An extended node list is defined, with each nodal value repeating itself once:

$$\hat{x}_{2j} = \hat{x}_{2j+1} = x_j,\ 0 \le j \le M-1 \Leftrightarrow \hat{x}_j = \begin{cases} x_{j/2} & \text{even } j \\ x_{(j-1)/2} & \text{odd } j \end{cases},\ 0 \le j \le 2M-1 \qquad \text{(III-1)}$$

The finite dimensional representation of $f(x)$ is made as

$$f(x) = \sum_{j=0}^{2M-1} \psi_j \Psi_j(x) \qquad \text{(III-2)}$$

where for even j

$$\Psi_j(x) = \begin{cases} \Psi_j^-(x) = \dfrac{(x-\hat{x}_{j-2})^2[2(\hat{x}_j - x) + (\hat{x}_j - \hat{x}_{j-2})]}{(\hat{x}_j - \hat{x}_{j-2})^3}, & \hat{x}_{j-2} \le x < \hat{x}_j \\ \Psi_j^+(x) = \dfrac{(x-\hat{x}_{j+2})^2[2(x - \hat{x}_j) + (\hat{x}_{j+2} - \hat{x}_j)]}{(\hat{x}_{j+2} - \hat{x}_j)^3}, & \hat{x}_j \le x < \hat{x}_{j+2} \\ 0,\ x < \hat{x}_{j-2}, x > \hat{x}_{j+2} \end{cases} \qquad \text{(III-3)}$$

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and for odd j

$$
\Psi_j(x) = \begin{cases} \Psi_j^-(x) = \dfrac{(x-\hat{x}_{j-2})^2(x-\hat{x}_j)}{(\hat{x}_j-\hat{x}_{j-2})^2}, & \hat{x}_{j-2} \le x < \hat{x}_j \\[2ex] \Psi_j^+(x) = \dfrac{(x-\hat{x}_{j+2})^2(x-\hat{x}_j)}{(\hat{x}_{j+2}-\hat{x}_j)^2}, & \hat{x}_j \le x < \hat{x}_{j+2} \\[2ex] 0, \; x < \hat{x}_{j-2}, x > \hat{x}_{j+2} & \end{cases} \tag{III-4}
$$

Since $x_{j^*} < x \le x_{j^*+1} \Leftrightarrow \hat{x}_{2j^*} < x \le \hat{x}_{2j^*+3}$, at most 4 terms directly contribute to approximating $f(x)$

$$
\begin{gathered} f(x) = \sum_{j=2j^*}^{2j^*+3} \psi_j \Psi_j(x) = \\ \psi_{2j^*}\Psi_{2j^*}^+(x) + \psi_{2j^*+1}\Psi_{2j^*+1}^+(x) + \psi_{2j^*+2}\Psi_{2j^*+2}^-(x) + \psi_{2j^*+3}\Psi_{2j^*+3}^-(x) \end{gathered} \tag{III-5}
$$

Let us express (III-5) in a more compact way. If $j\bullet$ is the smallest (even) nodal value such that $\hat{x}_{j\bullet} \le x < \hat{x}_{j\bullet+2}$, then

$$
f(x) = \psi_{j\bullet}\Psi_{j\bullet}^+(x) + \psi_{j\bullet+1}\Psi_{j\bullet+1}^+(x) + \psi_{j\bullet+2}\Psi_{j\bullet+2}^-(x) + \psi_{j\bullet+3}\Psi_{j\bullet+3}^-(x) \tag{III-6}
$$

For the optimal hedging application in this paper, a discrete (M in number) set of values of the reference asset, at time-step k, will be chosen as nodal locations for the finite dimensional representation for the option value and the hedging parameter.

The basis functions (III-3)-(III-4) are defined such that the values of the parameters multiplying them provide directly provide the values of f and df/dx:

$$
\psi_j = \begin{cases} f(\hat{x}_j) & \text{even } j \\[2ex] \dfrac{df(\hat{x}_j)}{dx} & \text{odd } j \end{cases} \tag{III-7}
$$

This property is useful to examine departures from delta hedging because the derivative of the option value with respect to the asset price is directly accessible.

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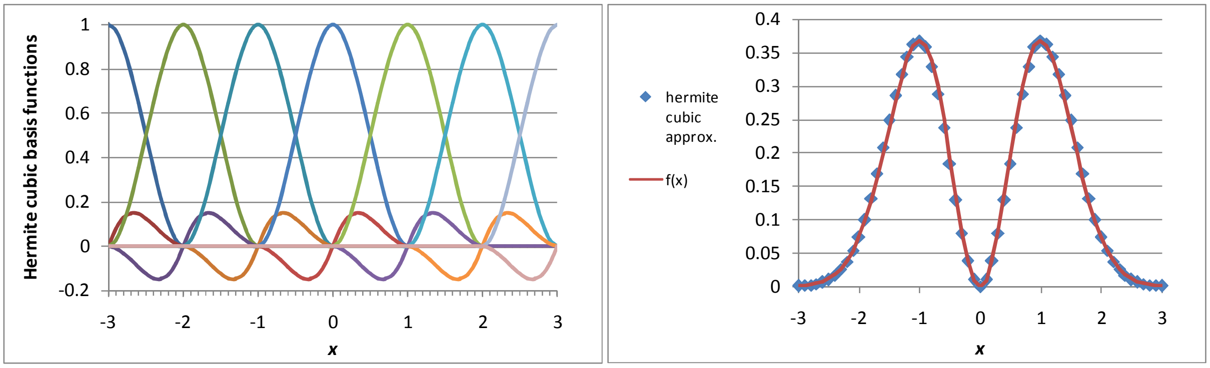

Figure III-1. Piecewise Hermite cubic polynomial representation of $f(x) = x^2\exp[-x^2]$ over [-3,3] using 7 nodes . The 14 basis function and numerical valuations of the piecewise Hermite representation and direct evaluations of the function are shown in the plots.

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