Abstract
In this paper we present the sequence of the k-Jacobsthal-Lucas numbers that generalizes the Jacobsthal-Lucas sequence. We establish an explicit formula for the term of order n, the well-known the Binet’s formula for k-Jacobsthal-Lucas numbers and obtain some properties and giving the generating function for the k-Jacobsthal-Lucas sequences.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, k- Jacobsthal numbers, k- Jacobsthal –Lucas numbers, Binet’s formula, Generating Functions.

1. INTRODUCTION:

The Fibonacci sequence is an inexhaustible source of many interesting identities. It is one of the most famous numerical sequences in mathematics and constitutes an integer sequence. Fibonacci numbers are a popular topic for mathematical enrichment and popularization. The Fibonacci sequence is famous for possessing wonderful and amazing properties. The Fibonacci appear in numerous mathematical problems. Fibonacci composed a number text in which he did important work in number theory and the solution of algebraic equations. The book for which he is most famous is the “Liber abaci” published in 1202. 

Falcon and Plaza [4] show the relation between the 4TLE partition and the Fibonacci numbers, as another example of the relation between geometry and numbers. The use of the concept of antecedent of a triangle is used to deduce a pair of complex variable functions. These functions, in matrix form, allow us too directly and in an easy way, present many of the basic properties of some of the best known recursive integer sequences, like the Fibonacci numbers and the Pell numbers. Djordjevic [5] introduce and investigate some properties and relations involving sequences of numbers \( F_{n,m}(r) \), for \( m = 2; 3; 4 \), and \( r \) is some real number. These sequences are generalizations of the Jacobsthal and Jacobsthal Lucas numbers. Arunkumar, Kannan and Srikant [8] determine a relation between Jacobsthal number and prime Jacobsthal number for twin prime numbers, and an inequality is found between Fibonacci Jacobsthal numbers. Koken and Bozkurt [3] deduce some properties and Binet like formula for the Jacobsthal number by matrix method. Falcon and Plaza [4, 7] introduced a new generalization of the classical Fibonacci sequence and Catrino [6] introduced generalization of Pell sequence. It should be noted that the recurrence formula of these numbers depends on one real parameter \( k \). Jhala, Sisodiya and Rathore [2] define the k-Jacobsthal number in an explicit way, and many properties are proved by easy arguments for the k-Jacobsthal number. Horadam [1] define the Jacobsthal \( \{ J_n \} \) and the Jacobsthal-Lucas sequences \( \{ J_n \} \).

The Jacobsthal sequence \( \{ J_n \} \) is define recurrently by
\[
J_{n+1} = J_n + 2J_{n-1}; \text{ for } n \geq 1 \text{ with initial condition } J_0 = 0, \quad J_1 = 1. \quad (1.1)
\]

The Jacobsthal-Lucas sequence \( \{ J_n \} \) is define recurrently by
\[
J_{n+1} = J_n + 2J_{n-1}; \text{ for } n \geq 1 \text{ with initial condition } J_0 = 2, \quad J_1 = 1. \quad (1.2)
\]
Jhala [2] define the k-Jacobsthal sequence \( \{ J_{k,n} \} \) is defined recurrently by
\[
J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}; \text{ for } n \geq 1 \text{ with initial condition } J_{k,0} = 0, \quad J_{k,1} = 1. \tag{1.3}
\]
In this paper we focused on k-Jacobsthal-Lucas number and obtain Binet formula for the k-Jacobsthal-Lucas number. Moreover, we derived some identities and generating function for the k-Jacobsthal-Lucas.

2. k-JACOBSTHAL LUCAS NUMBER:
For any positive real number k, the k-Jacobsthal-Lucas sequence is defined recurrently by
\[
j_{k,n+1} = k j_{k,n} + 2 j_{k,n-1}; \text{ for } n \geq 1 \tag{2.1}
\]
with initial condition \( j_{k,0} = 2, \quad j_{k,1} = k \). \tag{2.2}

Next we find the explicit formula for the term of order \( n \) of the k-Jacobsthal-Lucas sequence using the well-known results involving recursive recurrences. Consider the following characteristic equation, associated to the recurrence relation (2.1),
\[
r^2 - kr - 2 = 0 \tag{2.3}
\]
has two distinct roots \( r_1 \) and \( r_2 \) being \( r_1 = \frac{k + \sqrt{k^2 + 8}}{2} \) and \( r_2 = \frac{k - \sqrt{k^2 + 8}}{2} \), where \( k \) is a real positive number. Since \( k > 0 \), then \( r_2 < 0 < r_1 \) and \( |r_2| < |r_1| \). Also we obtain \( r_1 + r_2 = k, \quad r_1 r_2 = -2 \), and \( r_1 - r_2 = \sqrt{k^2 + 8} \).

3. PROPERTIES OF K-JACOBSTHAL LUCAS NUMBER:
3.1 Binet’s Formula
In the 19th century, the French mathematician Binet derived two remarkable analytical formulas for the Fibonacci and Lucas numbers [9]. In our case, Binet’s formula allows us to express the k-Jacobsthal Lucas numbers in function of the roots \( r_1 \) & \( r_2 \) of the following characteristic equation, associated to the recurrence relation (2.1),
\[
r^2 - kr - 2 = 0
\]

Theorem 1: (Binet’s Formula)
The nth k-Jacobsthal-Lucas number is given by
\[
j_{k,n} = r_1^n + r_2^n \tag{3.1}
\]
where \( r_1, r_2 \) are the roots of the characteristic equation (2.3) and \( r_1 > r_2 \).

Proof: Since the characteristic equation has two distinct roots, the sequence
\[
j_{k,n} = c_1 r_1^n + c_2 r_2^n \tag{3.2}
\]
is the solution of the equation (2.1). Giving to \( n \) the values \( n = 0 \) and \( n = 1 \) and solving this system of linear equations, we obtain a unique value for \( c_1 \) and \( c_2 \). So, we get the following distinct values, \( c_1 = 1 \) and \( c_2 = 1 \). Now, using (3.2), we obtain (3.1) as required.

3.2 Convoluted Product Identity

Theorem 2: (Convoluted product identity)
\[
j_{k,m+n} = j_{k,n+1} J_{k,m} + 2 j_{k,n} J_{k,m+1} \tag{3.3}
\]

Proof: Taking R.H.S. and applying the Binet’s formula for both k-Jacobsthal numbers [2] and k-Jacobsthal Lucas numbers (3.1).
\[ j_{k,n+1}j_{k,m} + 2j_{k,n}j_{k,m-1} = \left(r_1^{n+1} + r_2^{n+1}\right)\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right) + 2\left(r_1^n + r_2^n\right)\left(\frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}\right) \]

\[ = \frac{1}{r_1 - r_2}\left\{r_1^{n+1} + r_2^{n+1}\right\}\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right) + 2\left(r_1^n + r_2^n\right)\left(\frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}\right) \]

\[ = \frac{1}{r_1 - r_2}\left\{r_1^{m+n} \left(r_1 + 2r_1^{-1}\right) - r_2^{m+n} \left(r_2 + 2r_2^{-1}\right) - r_1^n r_2^m \left(r_1 + 2r_1^{-1}\right) + r_2^n r_1^m \left(r_2 + 2r_2^{-1}\right) \right\} \]

\[ = \left(r_1^{m+n} + r_2^{m+n}\right) \left(r_1 - r_2\right) - r_2^{m+n} \left(r_2 - r_1\right) \]

\[ = j_{k,m+n} \quad \text{[From (3.1)]} \]

That is \( j_{k,m+n} = j_{k,n+1}j_{k,m} + 2j_{k,n}j_{k,m-1} \)

### 3.3 Summation Formula

**Theorem 3 (Summation formula)**

Summation formula for k-Jacobsthal Lucas number is given by

\[ \sum_{i=1}^{n} j_{k,i} = \frac{j_{k,n+1} + 2j_{k,n} - k - 4}{k + 1} \quad (3.4) \]

**Proof:** Using the Binet formula (3.1) and the fact that \( r_1r_2 = -2 \) we get

\[ \sum_{i=1}^{n} j_{k,i} = \sum_{i=1}^{n} \left(r_1^i + r_2^i\right) \]

\[ = \frac{r_1 - r_1^{n+1}}{1 - r_1} + \frac{r_2 - r_2^{n+1}}{1 - r_2} \]

\[ = \frac{r_1 - r_1 r_2 - r_1^{n+1} + r_1^{n+1} r_2 + r_2 - r_2 r_1 - r_2^{n+1} + r_2^{n+1} r_1}{(1 - r_1)(1 - r_2)} \]

\[ = \frac{-(r_1^{n+1} + r_2^{n+1}) + r_1 r_2 (r_1^n + r_2^n) + (r_1 + r_2) - 2r_1 r_2}{(1 - r_2 - r_1 + r_1 r_2)} \]

\[ \sum_{i=1}^{n} j_{k,i} = \frac{j_{k,n+1} + 2j_{k,n} - k - 4}{k + 1} \]

### 3.4 Relation Between The K-Jacobsthal And K-Jacobsthal Lucas Number

**Theorem 4 (First Relation)**

\[ j_{k,n}^2 = \left(k^2 + 8\right)j_{k,n} + 4(-2)^n \quad (3.5) \]

**Proof:** Taking Right hand side and using the Binet formula (3.1)
\[(k^2 + 8)J_{k,n}^2 + 4(-2)^n = \left(k^2 + 8\right)\left(\frac{r_i^n - r_2^n}{r_1 - r_2}\right)^2 + 4(-2)^n\]

\[= r_1^{2n} + r_2^{2n} - 2r_1^n r_2^n + 4r_1^n r_2^n\]

\[= \left(r_1^n + r_2^n\right)^2\]

\[= j_{k,n}^2\]

That is \[(k^2 + 8)J_{k,n}^2 + 4(-2)^n = j_{k,n}^2\]

**Theorem 5 (Second Relation)**

Between the k-Jacobsthal Lucas number and the k-Jacobsthal number it is verified

\[j_{k,n} = 2J_{k,n-1} + J_{k,n+1} \text{ for } n \geq 1\] (3.6)

**Proof:** By induction, if \(n = 1\), then \(2J_{k,0} + J_{k,2} = k = j_{k,1}\)

Formula (3.6) is true for \(n = 1\)

let us suppose that formula is true until \(n - 1\) then

\[j_{k,n-2} = 2J_{k,n-3} + J_{k,n-1}\]

\[j_{k,n-1} = 2J_{k,n-2} + J_{k,n}\] (3.7)

So

\[j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}\]

\[j_{k,n} = k\left(2J_{k,n-2} + J_{k,n}\right) + 2\left(2J_{k,n-3} + J_{k,n-1}\right)\] [From (2.1)]

\[= 2\left(kJ_{k,n-2} + 2J_{k,n-3}\right) + \left(kJ_{k,n} + 2J_{k,n-1}\right)\]

\[= 2J_{k,n-1} + J_{k,n+1}\]

That is

\[j_{k,n} = 2J_{k,n-1} + J_{k,n+1}\]

**Theorem 6 (Third Relation)**

For \(n \in N\), \(j_{k,n}J_{k,n} = J_{k,2n}\) (3.8)

Proof: we know that convolution product identity for k-Jacobsthal number is given by

\[J_{k,n+m} = J_{k,n+1} J_{k,m} + 2J_{k,n} J_{k,m-1}\]

Now let \(m = n\)

\[J_{k,2n} = J_{k,n+1} J_{k,n} + 2J_{k,n} J_{k,n-1}\]

\[= \left(J_{k,n+1} + 2J_{k,n-1}\right)J_{k,n}\]

\[= \frac{1}{k}\left(J_{k,n+1} + 2J_{k,n-1}\right)\left(J_{k,n+1} - 2J_{k,n-1}\right)\]

\[= j_{k,n}J_{k,n}\] [From (1.3) & (3.6)]

### 3.5 Catalan’s Identity

**Theorem 7: (Catalan’s identity)**
\[ j_{k,n+r} - j_{k,n} = (-2)^{n-r} \left\{ j_{k,r}^2 - 4(-2)^r \right\} \]  
\( (3.9) \)

**Proof:** Using the Binet formula (3.1) and the fact that \( r_2^r = -2 \) we obtain the identity (3.9).

### 3.6 Cassini’s Identity

**Theorem 8:** (Cassini’s identity)

\[ j_{k,n-1}j_{k,n+1} - j_{k,n}^2 = (-2)^{n-1}(k^2 + 8) \]  
\( (3.10) \)

**Proof:** for \( r = 1 \) in Catalan’s identity, we obtain the Cassini’s identity for the \( k \)-Jacobsthal-Lucas sequence.

### 3.7 d’Ocagne’s Identity

**Theorem 9:** (d’Ocagne’s identity)

If \( m > n \) then

\[ j_{k,m}j_{k,n+1} - j_{k,n}j_{k,m+1} = (-2)^n \sqrt{k^2 + 8} \left\{ j_{k,m-n} - 2 \left( \frac{k + \sqrt{k^2 + 8}}{2} \right)^{m-n} \right\} \]  
\( (3.11) \)

**Proof:** Once more, using the Binet’s formula (3.1), the fact that \( r_2^r = -2 \) and \( m > n \), we obtain the identity (3.11).

### 3.8 Generating Function For The \( k \)-Jacobsthal-Lucas Number

Next we shall give the generating functions for the \( k \)-Jacobsthal-Lucas sequence. We will show that \( k \)-Jacobsthal-Lucas sequence can be considered as the coefficients of the power series of the corresponding generating function.

**Theorem 8:** Generating function for \( k \)-Jacobsthal Lucas number is given by

\[ \mathcal{Z}_k(x) = \frac{2-kx}{1-kx-2x^2} \]  
\( (3.12) \)

**Proof:** Let us suppose that the \( k \)-Jacobsthal-Lucas numbers of order \( k \) are the coefficients of a power series centered at the origin, and let us consider the corresponding analytic function \( \mathcal{Z}_k(x) \) defined by

\[ \mathcal{Z}_k(x) = j_{k,0} + j_{k,1}x + j_{k,2}x^2 + j_{k,3}x^3 + \ldots + j_{k,n}x^n \]  
\( (3.13) \)

and called the generating function of the \( k \)-Jacobsthal-Lucas numbers.

Using the initial conditions (2.2), we get

\[ \mathcal{Z}_k(x) = 2 + kx + \sum_{n=2}^{\infty} j_{k,n}x^n \]  
\( (3.14) \)

Now from (2.1) we can write (3.14) as follows

\[ \mathcal{Z}_k(x) = 2 + kx + \sum_{n=2}^{\infty} (kj_{k,n-1} + 2j_{k,n-2})x^n \]  
\( (3.15) \)

Consider the right side of the equation (3.15) and doing some calculations, we obtain that

\[ 2 + kx + \sum_{n=2}^{\infty} (kj_{k,n-1} + 2j_{k,n-2})x^n = 2 + kx + k \sum_{n=2}^{\infty} j_{k,n-1}x^n + 2 \sum_{n=2}^{\infty} j_{k,n-2}x^n \]

\[ = 2 + kx + k \sum_{n=2}^{\infty} j_{k,n-1}x^{n-1} + 2x^2 \sum_{n=2}^{\infty} j_{k,n-2}x^{n-2} \]  
\( (3.16) \)
Consider that $j = n - 2$ and $p = n - 1$. Then (3.16) can be written by

$$\sum_{p=0}^{\infty} j_{k,p} j_{j,p} + j_{k,0} x^p + 2x^2 \sum_{j=0}^{\infty} j_{k,j} x^j$$

Therefore

$$\mathcal{J}_j(x) = 2 - kx + kx \mathcal{J}_0(x) + 2x^2 \mathcal{J}_1(x)$$

which is equivalent to

$$\mathcal{J}_j(x) \left( 1 - kx - 2x^2 \right) = 2 - kx$$

and then the ordinary generating function of the $k$-Jacobsthal-Lucas sequence can be written as

$$\mathcal{J}_k(x) = \frac{2 - kx}{1 - kx - 2x^2}$$

4. CONCLUSION:

In this paper, $k$-Jacobsthal Lucas sequence is introduced. Some standard identities of $k$-Jacobsthal Lucas sequence have been obtained and derived using generating function and Binet’s formula.

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6. REFERENCES: