Solving Polynomial Inequalities with GeoGebra: Opportunities of Visualization and Multiple Representations

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Abstract: The main purpose of the present study is to investigate the ways students' instrumentation of Computer Algebra System (CAS) can help promote their algebraic reasoning while solving polynomial inequalities. In addition, the relation between students' CAS techniques and paper-and-pencil (P&P) techniques are explored, together with the difficulties that students may face as they apply these techniques. Research participants are 33 tenth graders at a private mixed gender school in Mount-Lebanon, distributed among nine homogenous groups, five of which are selected as focus groups. The study is qualitative in nature. Data is collected from pretests, students' written solutions of four instructional activities, laptop screen recordings, video recordings of whole-class discussions, and audio recorded interviews with students in the focus groups. The findings of the study show that students' lack of prerequisite knowledge of the topic of functions and their low level of familiarity with GeoGebra software are determinant factors that hinder these students' instrumentation of CAS and hence their reasoning processes as well as their implementation of the solving techniques. High and middle-achieving students' solving techniques acquired little epistemic and some pragmatic values, whereas low achieving students' solving techniques acquired heuristic values.

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1. Introduction:

'Inequalities', in general, and polynomial inequalities in particular, are important topics that interweave with most mathematical topics. According to Tanner (1962), they are "the most important tool in the workshop of the mathematician and the most responsible for shaping mathematics as we now know it" (p. 161); further, Alsina and Nelsen (2009) contend that inequalities have had a long distinguished role in the evolution of mathematics. However, in order to appreciate the value of the aforementioned, it is necessary to perform certain operational manipulations and consequently establish meaning and relationships. This leads to another function that is inherent in such a development or the reasoning ability. According to Yackel and Hanna (2003), reasoning can have many functions including verification, explanation, systematization, discovery, communication, construction of theory, and exploration.

Reasoning as a "foundation of mathematics" (Stacey & Vincent, 2009, p. 271) had been used by Jones (2000) to mean "making reasonably precise statements and deductions about properties and relationships" (p. 69). As for algebraic reasoning, Kaput and Blanton (2005) indicate that it includes students' ability to "generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation" (p. 99).

The authors have noticed that when solving linear, quadratic inequalities, and some kinds of higher order factorable polynomial inequalities, in a paper-and-pencil environment, students seemingly engage in calculations without resorting to reason to find solutions. They tend to believe that reasoning is, most of the time, related to geometry, consequently, no reasoning is needed when working with algebraic activities.

1.1. Problem definition:

The Lebanese curricula do not stipulate that linear, quadratic and some higher order factorable polynomial inequalities be full-fledged topics, but are taught as minor topics or as prerequisites for other topics. Moreover, polynomial inequalities are treated as purely algebraic and abstract topics. At the secondary level, textbooks required for Lebanese public schools and used by most teachers, introduce the said topics to students before the latter are familiar with graphing functions. Here, graphs are used as tertiary aids or as aims for their own sake but not to add much to students' conceptual understanding; most of these exercises require drill work with lengthy and tedious algebraic calculations and construction of sign tables. This inconsistency leads students to lose interest and to find it difficult to understand the topic of polynomial inequalities. Yet, Hussain, Buus, Kiros, Wichmann, Selvarajah, and Ahmad (2007) contend that Computer Algebra System (CAS) applications are useful to support students' learning ability. CAS are "mathematical applications developed with the purpose of solving mathematical problems which are too difficult or even impossible to solve by hand. Modern versions of CAS applications are known for their rather large set of features such as support for graphical representations of results, symbolic manipulation, big-integer calculations, and complex-number arithmetic" (p. 45).

Hence, it is conjectured that the Computer Algebra System (CAS), as a tool package with different views and mathematical environments, can offer a suitable medium for solving polynomial inequalities as it can free students from drill work. According to Ruthven (2002), CAS allows "instrumenting graphic and symbolic reasoning … and influences the range and form of the tasks and ***techniques*** experienced by students" (p. 275). As a result of working with a series of similar tasks, students develop a "structured set of the generalizable characteristics of artifact utilization activities". This set which forms a "stable basis" for students' activity was defined by Verillon and Rabardel (1995) as *utilization schemes*. The process of developing *instrumented techniques* and *utilization schemes* is defined as *instrumentation* or *instrumental genesis* (Drijvers, 2003).

At this point, some definitions are necessary to add clarity to the aforementioned statement. Exhibit 1 depicts the necessary definitions to clarify what CAS allows.

Exhibit 1. Supporting definitions

A technique: It is defined as "a manner of solving a task" (Artigue, 2002, p. 248) which, according to Lagrange (2005), when "related to the tool that makes them possible" becomes an "instrumented technique" (p.132). Techniques can be elementary, such as the direct application of one single command or a gesture or, according to Drijvers (2003), can be composed of a set of gestures.

Gestures: They are taken to mean the "idiosyncratic spontaneous movements of the hands and arms accompanying speech" (Neill, 1992, p. 37). Trouche (2003) posits the use of gestures with the *operative invariants* that guide their form, or the "instrumented action schemes" (p. 7).

Operative invariants: They are defined as the "implicit knowledge contained in the schemes" (Trouche, 2004, p. 286).

Schemes: According to Vergnaud, schemes are "the invariant organization of the behavior" (as cited in Guin & Trouche, 2002, p. 205). Consequently, gestures, according to Trouche (2005a), form the "observable part of an instrumented action scheme" (p.151).

Limitations of CAS have led to the demand for a strict syntax when using the software commands. If students fail to adapt CAS's conventions/notations and techniques to their existing schemes, difficulties will hinder the emergence of students' instrumentation schemes. The aforementioned difficulties or obstacles are "technical and/or conceptual barriers encountered in the CAS environment that prevent students from carrying out the instrumentation scheme they had in mind" (Drijvers, 2000, p. 195). Therefore, instructors have to pay attention to the fact that students need to be well aware of CAS potential and the specific conventions/notations to integrate such software in their learning in the classroom. Consequently, taking the aforementioned into consideration, the next section delineates the rationale of this study and the particular technical details needed for the experiments on hand.

1.2.The rationale for the study:

This research aims to explore the development of grade 10 students' thinking and solving techniques in a CAS environment while learning "even-powered and odd-powered" polynomial inequalities. From the historical point of view, since inequalities are associated with the order, they arose as soon as people started using numbers, making measurements, and later, finding approximations and bounds. Students, according to Sangwin (2015), at the International Baccalaureate Higher Level (HL) Mathematics, are assumed to be able to express the solution set of a linear inequality on the number line and in set notation and are also expected to know the properties of order relations.

The importance of inequalities in the classroom arises as soon as the ordering of numbers (in the primary grades) and solving linear inequalities by algebraic and graphical methods (in middle grades) is considered. In-plane and solid geometry, inequalities appear naturally when comparing measures (lengths, areas, volumes, etc…), in determining the existence or non-existence of particular figures, and solving optimization problems.

More specifically, this study seeks to inspect how the instrumentation of CAS can help students promote their algebraic reasoning while solving polynomial inequalities. The study also investigates the mutual effect of P&P (paper-and-pencil) techniques and CAS techniques and the possibility of transfer of techniques between the two environments. From one side, students can try to apply the P&P techniques while working with CAS or try to adapt the CAS techniques while working in a P&P environment.

The technical (or conceptual) difficulties that students encounter are also investigated.

This research highlights the issues of CAS integration in the Lebanese curriculum and how could it contribute to filling the gap in the Lebanese research structure or even in the research structure at the regional level. Moreover, the study highlights the matter of sequencing of the topics within the Lebanese mathematics curriculum, taking the topics of polynomial inequalities and functions as an example.

2. Literature Review:

Earlier studies pointed to the influence of CAS use on building students' mathematical knowledge (Guin & Trouche, 2002) and thinking (Drijvers & Graveneijer, 2005). Additional studies explored the difficulties that students face when working in CAS environments (Drijvers, 2000, 2003), while other studies investigated how working with CAS affects students' mathematical reasoning (Kramarski & Hirsch, 2003) and the techniques that they use (Kieran & Drijvers, 2006).

2.1. Knowledge and technology:

Teachers' knowledge, according to Shulman (1986), was defined and tested, as in California Teachers Examination, in terms of subject matter, pedagogical skill, some aspects of physiology, knowledge of theories and methods of teaching. A research-based view emerged in the 1980s, where knowledge of subject matter was nearly substituted by knowledge of organization and management of classrooms as a necessary asset and skill for an expert pedagogue (Berliner, 1986).

At the time of Shulman, technology's relationship to pedagogy and content had not yet been discussed. After the 1980s, technologies, mainly referring to digital computers and computer software, came to the forefront of educational discourse. The view of '*knowledge of technology*' as being isolated from knowledge of pedagogy and content became inappropriate (Mishra & Koehler, 2006).

Today with the widespread use of modern technology, it became inevitable that most students have to deal with this technology. Moreover, according to Hughes (2005), teachers learning about technology from a content perspective are more prone to use it to support content learning. New technologies can offer opportunities for widening the scope of mathematical concepts that students are able to discover (Dana-Picard, 2005). New technologies can, according to Roe, Pratt, and Jones (2003), foster the "genesis of connections with complex scientific ideas" (p. 1099).

2.2 GeoGebra (GG) and Computer Algebra Systems (CAS):

This research concentrates on the instrumentation of CAS implemented by the tenth graders while solving polynomial inequalities. The CAS, used in this study, is a part of GeoGebra that was invented in the early 2000s. GeoGebra is a community-supported open-source mathematics learning environment that integrates multiple dynamic representations, various domains of mathematics and a rich variety of computational utilities for modeling and simulations. It is a new, cost-free and very innovative technology that can be used to support the progressive development of mental models appropriate for solving complex problems involving mathematical relationships. The software originated in the Master's Thesis Project of Markus Hohenwarter, at the University of Salzburg, in 2002.

It was designed to combine features of dynamic geometry software and computer algebra systems in a single, integrated and easy-to-use system for teaching and learning mathematics (Hohenwarter & Preiner; cited in Hohenwarter & Lavicza, 2011). The GeoGebra project represents a form of "synergy or concerted effort between technology and theory, individual inventions and collective participation, local experiments and global applications" (Bu & Schoen, 2011). As indicated by Lingguo Bu, Spector and Haciomeroglu (2011), a synthesis of the theoretical frameworks including Realistic Mathematics Education (RME), Model-Facilitated Learning (MFL), and Instrumental Genesis (IG) can be used when teaching and learning mathematics in a GeoGebra environment.

The Lebanese curriculum considers the "calculator with memory" as a tool for performing calculations in primary classes, and hints at "the possibility of using the computer" as "technological novelties which will have benefits on the formation" (CERD, 2007). Within the statement of the objectives of this curriculum, only a shy indication of technology use was made without specifying clear methods and plans for integrating this technology.

Technology integration into teaching and learning is not based, most of the time, on awareness and preparation, but on teachers' personal perceptions and views. New technologies offer the possibility of approaching problem-solving in novel ways that depend on visualization. In teaching mathematics, visualization is essential to develop intuition and to clarify concepts. It is believed that visualization can be a powerful tool for better understanding of some basic mathematical facts as is the case of drawing pictures/figures to solve the problem. Drawing figures and visualization open new avenues to creative ways of thinking and teaching. Thus, GeoGebra is believed to be a tool package, including CAS, that can furnish a suitable environment where visualization and graphs are devised to solve polynomial inequalities.

2.3 Inequalities:

Inequalities, associated with the order, "arose as soon as people started using numbers, making measurements, and later, finding approximations and bounds" (Alsina & Nelsen, 2009, p. xvi). The Hindu and the Chinese knew some kinds of inequalities as geometric facts (Fink, 2000). After that, nothing much happened until Newton and Cauchy, a century later. Algebraic processes have not been expressed by symbols for a long time and a mathematical expression was initially oral. Ancient inequalities, too, were expressed by verbal registers (Bagni, 2005).

In this respect, Lakoff and Núñez note that: "It may be hard to believe, but for two millennia, up to the 16th century, mathematicians got by without a symbol for equality" (Lakoff & Núñez; cited in Bagni, 2005). This agrees with Tanner (1962) when he says: "It is fascinating to observe how the Greeks, without any symbolism to help them, were able to grasp so thoroughly the implication and power of inequalities" (p. 161). To express that one area is larger than another, Euclid, for example, used the words: "falls short of" or "is in excess of" but no arithmetic of inequalities for numbers is indicated by any of the ancient traditions (Fink, 2000, p. 120). Also, Alsina and Nelsen indicate that:

"The symbol (=) for equality appears to have been introduced by Robert Record (c. 1510–1558) in his book The Whetstone of Witte, published in 1557. This symbol did not appear in print again until 1618, but soon thereafter replaced words commonly used to express equality, such as aequales (often abbreviated aeq), esgale, faciunt, ghelijck, and gleich. The symbols > and < to denote strict inequality appeared a few years later, in The Analytical Arts by Thomas Harriot (1560–1621), published in 1631. (…) Harriot states the meaning for > and < quite clearly: Signum majoritatus ut a > b significet a majorem quam b, and Signum minoritatus ut a < b significet a minorem quam b (a > b means "a" is larger than "b", and a < b means "a" is smaller than "b"). (…) Nevertheless, 1631 is the birth date for > and <. Pierre Bouguer (1698–1758) used ≧ and ≦ in 1734, while John Wallis (1616–1703) used similar notation but with the bars above the inequality symbols" (Alsina & Nelsen, 2009, p. xviii).

Today, inequalities are present in nearly every branch of mathematics, and the study of inequalities has become a field by itself. They interweave on various mathematical topics including algebra, trigonometry, linear programming and the investigation of functions. Inequalities, according to Tsamir, Almog, and Tirosh (1998), also provide a complementary perspective to equations. Solving inequalities, according to Bagni (2005), used to be achieved by solving an equation that practically replaces the assigned inequality. Then the binding conditions that govern the accepted solutions of the considered equations were expressed by inequalities. Yet techniques for equation solving, when applied to inequalities, have led sometimes to wrong final solutions. However, inequalities can be solved using different strategies, including numerical and algebraic manipulations, drawing graphs, and using the number-line.

Despite recommendations by the National Council of Teachers of Mathematics (NCTM) to teach inequalities at all grade levels, these recommendations have received relatively little attention and are (especially, polynomial inequalities) usually discussed in the upper grades of the secondary school. The Lebanese curriculum tackles the topic of 'inequalities' in a fragmented and inconsistent way across grade levels, even, without offering an explicit method for approaching the topic. To illustrate, inequalities are tackled explicitly in the eighth grade with a focus on linear inequalities. Later on, in the second year of the secondary education, the curriculum addresses the topic of "studying the sign of quadratic trinomials and some kinds of higher order factorable polynomial inequalities".

Curricular approach to teaching the topic of inequalities is, most of the time, procedural and theoretical. Besides, when teaching inequalities, teachers incorporate artifacts without enough lesson planning and adequate preparation of the aforementioned lesson, which creates difficulties for students. Difficulties, according to Tsamir, Almog and Tirosh (1998), include: incorrectly deducing signs of factors from sign of product / quotient, solving an equation instead of an inequality, multiplying / dividing by factors that are not necessarily positive, forming meaningless connections with quadratic roots, and solving the square of the given inequality.

2.4. Reasoning:

Though the term 'reasoning', as indicated by Yackel and Hanna (2003), is mostly used among mathematics educators without defining it, yet some mathematicians use the term in different contexts. Jones (2000), for example, identifies mathematical reasoning as "making reasonably precise statements and deductions about properties and relationships" (p. 69). Reasoning, according to Walle, Karp, and Bay-Williams (2013), can also be taken to mean "the logical thinking that helps us decide if and why our answers make sense" (p. 4). This is in line with Lithner (2008) who considers reasoning as "the line of thought adopted to produce assertions and reach conclusions in task solving" (p. 257). Mathematical reasoning can be communicated through drawing, writing, talking, and also by mixing natural language and algebraic expressions.

According to Greenes and Findell (1999), mathematical reasoning develops when students become able to interpret algebraic equations using pictorial, graphics, and symbolic representations. Algebraic reasoning - a particular form of mathematical reasoning - is the "process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (Kaput & Blanton, 2005, p. 99). The algebraic reasoning is important because it drives students' understanding of mathematics beyond the result of specific calculations and the procedural application of formulas.

Algebraic reasoning can have many functions in mathematics, including verification, explanation, systematization, discovery, communication, construction of theory, and exploration (Yackel & Hanna, 2003).

This research explores how working in a CAS environment can contribute to the promotion of students' algebraic reasoning when solving polynomial inequalities. It is conjectured that working in CAS environment makes it easier for students to understand tasks than when working in a traditional, purely verbal, and algebraic context found in textbooks. CAS offers a potential for students to search for invariants and to propose corresponding conjectures. But many studies (for example, Drijvers, 2000) indicate also that students may encounter obstacles when working in such an environment.

2.5. Difficulties:

Obstacles encountered in a CAS environment are interpreted by Drijvers (2002) as an "unbalance of the conceptual and technical aspects of an instrumentation scheme" (p. 221). They may result from the black box character of the CAS (as it does not show the methods or techniques by which it obtains its solutions) or from the lack of congruence between the notations, language, and techniques in CAS and paper-and-pencil environments (Drijvers, 2003). Obstacles may be indicators of the difficulties that students encounter while developing conceptual understanding. Conceptual difficulties, according to Heck (2001), resulting from the differences between the algebraic representations found in the computer algebra environment and those encountered in traditional mathematics. While to Lagrange (2005), these difficulties may result from the difference between the techniques that students use within the computer algebra environment and those used in the traditional paper-and-pencil environment; a view shared by Drijvers and Gravemeijer (2005). To illustrate some of the difficulties while working with CAS, Drijvers (2000) enumerates the following ones:

- The difference between the algebraic representations provided by the CAS and those that students expect and conceive as 'simple'.
- The difference between numerical and algebraic calculations and the implicit way that this difference is dealt with CAS.
- The limitations of the CAS and the difficulty in providing algebraic strategies to help the students overcome these limitations.
- The inability to decide when and how computer algebra can be useful.
- The flexible conception of variables and parameters when implementing CAS.

2.6 Theoretical Framework (Theory of Instrumentation):

Vygotsky (1934) contends that "psychological tools master natural forms of individual behavior and cognition" (p. xxv). Before Vygotsky, the emphasis was on cognitive development resulting from interaction with material artifacts. In the year 1930, in addition to those of Vygotsky, attempts began to describe the psychological processes through which such a development could be envisaged. Based on Vygotsky's hypothesis which states that "artificial systems can extend man's cognitive capacities" (cited in Guin & Trouche, 1999), Verillon and Rabardel (1995) indicate that instruments bring about changes, "both structural and functional, in the subject's cognition" (p. 6). Artifacts, according to Verillon and Rabardel (1995), refer to all "objects of material culture to which an infant has access during his/her development" (p. 5).

Artifacts, regardless of their level of sophistication, have been a part of the human existence and activity and will continue to be so for centuries to come. Humans used available artifacts, elaborated and created others in a continuous process of use and invention, such as the ruler and the compass, artifacts from Euclid's era, formed the basis of the human mathematical activity and were at the roots of elementary geometry. Whichever is the artifact, the aim is not (only) the technical, but also the mathematical processes that evoke the properties of objects. The subject can use the artifact to carry out concrete actions; on the other hand, the artifact allows the formation of the subject's level of consciousness and cognition.

Trouche (2004) indicates that when specifying the artifact's "user and uses", the term 'tool' substitutes the term 'artifact' (p. 282). Tools are important for "sustaining" and "conditioning human activity" and even, according to Noss and Hoyles, "shape" the environment (cited in Trouche, 2004, p. 282). They can be used for modeling real-world problems, providing visualizations with interactive illustrations, and improving student's motivation and cognitive development. Tools, according to Doorman, Drijvers, Dekker, Van Den Heuvel-Panhuizen, De Lange, and Wijers (2007), "create new possibilities for problem solving in mathematics" (p. 415), and, according to Barzell, Drijvers, Maschietto and Trouche (2005), guide the choice of the solving strategies.

What is important about tools is the operating method which they impose on the user. The user, in turn, can "appropriate the tool for him/herself" (Verillon & Rabardel, 1995, p. 8) and "integrate it with his/her activity" through the process of "instrumental genesis" (Guin & Trouche, 2002, p. 205). The significance then is not the tool per se, nor the interaction of student and tool, but is the aims for which a tool is used and the schemes of its use. This evokes the distinction between the artifact and the instrument. The instrument is made up of the artifact and it's utilization schemes which are continuously updated and elaborated through the use of the artifact to accomplish certain tasks. Utilization schemes are defined by Verillon and Rabardel (1995) as "the generalizable characteristics of artifact utilization activities" (p. 12). As an intermediate universe (solution space) between the subject and object, the instrument will then be formed from the artifact, "either material or symbolic", and from "one or more associated utilization schemes" (Verillon & Rabardel, 1995, p. 13). The elaboration and evolution of instruments is a long and complex process that Rabardel names 'instrumental genesis' (Verillon & Rabardel, 1995).

Instrumental genesis, according to Artigue (2002), is directed "towards the artifact" and "towards the subject" and concerns the "emergence and evolution of utilization schemes, in which technical and conceptual elements co-evolve" (p. 250).

The two-dimensional relation between technical and conceptual aspects of a tool is reflected in the difference between instrumentation and instrumentalization (Drijvers & Gravemeijer, 2005); in other words, the effect is a two way coordinated process, including Instrumentation and Instrumentalization, which according to Trouche (2004) are identified as:

- Instrumentalization process: directed toward the artifact.
- Instrumentation process: directed toward the subject.

The instrumentalization process involves the emergence and development of the students' utilization schemes when performing a certain task. Performing a given task may take place at two levels. At the first level, students may give a technical solution of the task by using the artifact mechanically, i.e., repeating, in an automatized way, a set of instructions, without wondering why their solutions work. At this level, the justification of the correctness may not be at stake. At the second level, the solution becomes "meaningful" when it is justified and commented with reference to the properties and theorems in action.

According to a framework proposed by Pierce and Stacey (2004) for planning to teach and monitoring the progress of students using CAS for mathematics, the knowledge and skills for using CAS can be thought of "along with a continuum". At one extreme of the continuum, comes the "knowledge that relates only to the machine" involving "the technical aspect of the effective use of CAS", and at the other extreme comes the "mathematical knowledge". The technical aspect relates to students' ability to access the capabilities of CAS to achieve mathematical goals (Pierce and Stacey, 2004, p. 4). In between, however, there is a substantial body of knowledge that involves both mathematics and the machine.

The instrumentation process takes place through developing instrumented techniques and utilization schemes (Drijvers, 2003). A technique, according to Artigue (2002), is "a manner of solving a task" which is evaluated in terms of its pragmatic value by focusing on the "productive potential (efficiency, cost, field of validity)", and in terms of its epistemic value as it contributes to the "understanding of the objects" (p. 248). Moreover, according to Artigue (2002), a technique has a "heuristic role when it refers to the anticipations allowing to plan actions" (p. 259). This view, to the value of techniques, is shared by Lagrange (2005) who states that a technique plays a "pragmatic role when the important thing is to complete the task or when the task is a routine part of another task" or an "epistemic role by contributing to an understanding of the objects it handles" (p. 271). A technique, according to Lagrange (2005), becomes an instrumented technique when it is related to "the tool (…), to the mathematical domain and to the user's representations of both" (p. 132).

3. Methodology:

3.1.Research Questions :

This paper seeks to answer the following questions:

1. How can the instrumentation process of CAS promote students' algebraic reasoning while solving polynomial inequalities?
2. What is the relation between students' CAS techniques and paper-and-pencil (P&P) techniques when solving polynomial inequalities? What are the transfer and adaptation techniques between CAS environment and paper-and-pencil (P&P) environment when solving polynomial inequalities?
3. What difficulties (technical or conceptual) do students experience when using CAS (in the GeoGebra environment) to solve polynomial inequalities?

In fact, the study seeks to inspect how the instrumentation of CAS helps students to promote their algebraic reasoning. The study also investigates the mutual effect of P&P technique and CAS techniques and the possibility of transfer of techniques between the two environments. From one side, students will try to apply the P&P technique while working with CAS and from the other side they will try to adapt the CAS techniques while working in a P&P environment.

3.2. Participants, pretest and class sessions:

This research involves 33 tenth grade students, 14 to 16 years old, in a private, mixed-gender Lebanese school located in Mount-Lebanon. The students are distributed among nine homogeneous groups classified according to their achievement levels and teacher's recommendation during the previous and current years. High, middle and low achievers are referred to by the letters HA, MA and LA (representing the first letters of each of the two words) respectively. The symbol (G#) is used to indicate the group number, and the symbol (G#.#) is used to indicate the number of a particular student within that group. One group of high achievers (HAG), two groups of middle achievers (MAG1 and MAG2), and two groups of low achievers (LAG1 and LAG2) are chosen as focus groups and are indicated by the letter F.

Students were introduced to the general aims of the study, the content of the six class sessions (Exhibits 2a, 2b, 3a, 3b, 4a, 4b, 5a, and 5b), and the general guidelines for working in each session. The design principle '*structured problem solving*', by Stigler and Hiebert (1999), is followed in the design of the sessions of this study. Each student also received a '*Parental Consent Form*' and an '*Assent Form*' that were signed by the students and their guardians and collected by the teacher on the next day. Students were also told that they had to sit for a pretest (Exhibit 1) and that their solutions to the test and the four activities will be kept confidential, and that their solutions of the instructional activities will not be included in the calculation of their marks. Each group had a laptop equipped with GeoGebra (GG) and DVC.

In the first session (S1), the teacher, using a screen projector, introduced the *Algebra View* (AV), *Graphics View* (GV) and the *Input Bar*, from CAS. In the second session (S2), the teacher introduced Debut Video Capture (DVC) software. Four other sessions (S3, S4, S5 and S6) were each dedicated to implementing one of the instructional activities related to linear inequalities, even-powered polynomial inequalities, odd-powered polynomial inequalities and solving polynomial inequalities. After each session, students of the focus groups were interviewed so that they could reflect on their thinking and solution strategies, conjecturing, and reasoning. Students also reflected on the difficulties that they faced during the teaching sessions. The teacher distributed worksheets, managed time, and guided the whole class discussion; he/she then put together the results students found out at the end of the session. The teacher then collected students' written work and saved the computer files.

3.3. Data Analysis:

The data collected from pretests assisted in classifying students according to their achievement levels. Other forms of data included students' written solutions to four activities (Exhibits 2a,b; 3a,b; 4a,b; and 5a,b), DVC files, and audio recorded interviews. While analyzing data, the researchers looked for patterns and episodes of students' solutions that involved technical and conceptual elements. After that, it became feasible to deduce whether these elements are intertwined, and develop interactively and consequently would lead students to build utilization schemes (usage schemes and instrumented action schemes).

Moreover, students' solution strategies with the language, notations, symbols, and representations when implementing these strategies are explained and commented upon; this has helped to answer the first research question. In addition, the techniques that students used in their solutions of the different activities and the possible transfer of techniques from one environment to another (CAS and P&P) are highlighted. During the analysis of students' written work, the value of students' techniques (heuristic, pragmatic or epistemic) and their contribution to the process of instrumentation are indicated. The results of analyzing students' techniques, in this way, has helped answer the first research question about the instrumentation of CAS and also the second research question about the relation between CAS techniques and P&P techniques.

Moreover, saved screen recordings that included snapshots of laptop screens, accompanied by audio recordings of group's discussions (from DVC), have helped to determine the environments in which students worked, the tools and commands that they used to solve which parts of an activity, and for which purposes. This helped researchers to draw conclusions about how the instrumentation of CAS allowed students to promote (or not) their algebraic reasoning; these conclusions contributed to the answer of the first research question. The analysis also revealed information about the possible difficulties that students encountered when using CAS; this contributed to the answer of the third research question.

During the description phase of students' work, it was noticed that the work of the group MAG1 is similar, in most aspects, to the work of group MAG2; the same applied to the work of the groups LAG1 and LAG2. Consequently, it was decided that the description and analysis be limited to one middle-achiever group (MAG2) and one group (LAG2) of low achievers. The work of each of these groups is described and then analyzed in terms of the utilization schemes, of the value of their solving techniques, and the reasoning process that students went through while using the different tools to perform each category of tasks. Then, the possibility of transfer of techniques between the two environments is considered. Finally, the difficulties that students encountered during their work with the activities are described.

Here, and for the rest of the analysis, an *elementary technique* is taken, according to Drijvers (2003), to mean the "direct application of one single command" or "gesture" while a *composed technique* is taken to mean "a set of such gestures" (p.100).

To facilitate the reporting of the analysis of the results, each activity of the instructional unit is subdivided into minor tasks. A minor task is coded with a pair of numbers. To illustrate, the third minor task of the fourth activity is referred to as minor task 4.3. The first number (4) refers to the number of the activity, and the second number (3) refers to the part of that activity. Then, minor tasks of the same nature are classified into four categories:

i. The first category involves the investigation of the way the sign of a linear function varies.
ii. The second category involves the investigation of the graph behavior of polynomial functions.
iii. The third category involves the investigation of the way the sign of polynomial functions varies.
iv. The fourth category involves the solution of polynomial inequalities.

4. Results:

4.1. Addressing the research questions:

4.1.1.First research question:

How can the instrumentation process of CAS promote students' algebraic reasoning while solving polynomial inequalities?

The instrumentation theory focuses on the mediating role of tools (Artigue, 2002; Trouche, 2004); Lagrange (1999) defines mediation as "the use of properties of a given object to act on another" (p. 56). This research identifies two main types of mediation between the tools and the students, namely, the epistemic mediation and the pragmatic mediation. According to Rabardel and Bourmaud (2003), "An epistemic mediation contributes to understanding of the object involved, 'its properties, its evolutions in line with the subject's actions' (p. 668), whereas a pragmatic mediation concerns actions on the object, 'transformation, regulation management'" (p. 669).

Results reveal that, on three occasions, CAS played an epistemic mediation role and could help students to build elementary usage schemes.

On the first occasion, students (HAG) made a conjecture about the graph behavior of the function "$g(x) = x^4 - 1$," depending on visual perception of what they saw on the GV screen. Then, they used the *Zoom* tool (several times) to evaluate the reasonableness of their perception. The same procedure was repeated with another function "$h(x) = x^6 - 2x^5$" whose graph was apparently confounded with that of the x-axis near $x = 0$.

Students were alternatively using the *Zoom* tool, group discussion and making new conjectures, at the same time discovering new mathematical knowledge; for example, the compact nature of the coordinate axes as sets of real numbers. It can be inferred that this tool played an epistemic mediation role and helped students to build an elementary usage scheme oriented at investigating the graph behavior of a polynomial function at a particular point (minimum) of the graph. This elementary usage scheme involves technical aspects (using the *Zoom* tool by scrolling the wheel of the mouse inside the GV screen) and conceptual aspects (recognizing the compact nature of the coordinate axes as sets of real numbers).

On a second occasion, students (group HAG) used the *Zoom* tool to determine x-intercepts (which were not whole integers) while solving polynomial inequalities by the graphical method. They zoomed (several times), on the point where the graph cuts the x-axis so that the x-intercept appeared. This was similar to what is done when finding a reasonable approximate value of a zero of a function: performing several table iterations.

Again, the *Zoom* tool played an epistemic mediation role between the students and the task at hand and helped students to build another elementary usage scheme, oriented at determining the x-intercepts of a graph by using this tool. This scheme involves technical aspects (scrolling the wheel of the mouse inside the GV screen) and conceptual aspects (knowing that x-intercepts are points on the graph whose ordinates are zeros). Conceptual aspects (also involved calculating zeros of a function by performing several iterations to determine the solution of an equation).

On the third occasion, after describing the graph behavior of a number of odd-powered polynomial functions, students (group MAG2) were able to relate the form of the algebraic representation to the form of the graph to describe the graph behavior of a new function without using CAS to graph this function.

4.1.1.1. Instrumentation of CAS:

CAS tools allow students to move from descriptions (of graph behaviors) that depend on the visual perception of what appeared on the GV screen, to descriptions that depend on mental conceptions which are developed as a result of plotting several functions of the same type (odd-powered polynomial functions). For students, the algebraic representation became, according to Mariotti (2000), "a sign referring to a meaning" (p.36). Here, students shifted from a "perceptual to a conceptual model of understanding, that is, from being able to recognize, classify, and describe shapes of graphs to being able to define and deduce attributes and relationships among them" (Rivera, 2007, p.285). It can be concluded that CAS has played an epistemic mediation role, as the different CAS tools used helped the students to promote their conceptual understanding of the nature of the mathematical objects (polynomial functions) and to use this understanding to make new conjectures.

Accordingly, at the conclusion of the afore-stated three occasions, it can be inferred that the *Zoom* tool has played an epistemic mediation role and helped students to build usage schemes. Yet, each of these elementary usage schemes, could not be developed into instrumented action schemes because, according to (Drijvers, 2002), students were not able to perform the mental conceptions and technical actions "several times in similar situations, so that it becomes part of the 'repertoire' of the student" (p. 223).

4.1.1.2. Reasoning:

The available data reveals that students' reasoning is grounded on the connecting solutions that have shown consistency across different methods (for example, graphical and numerical). However, students' command process remained weak, with an avoidance of mathematical references. On the other hand, the primacy of numerical reasoning, in the P&P, did not prevent students from showing a preference for graphs. This may be because alternating between algebraic and graphical representations (in CAS) requires only one keystroke, rather than complex technical and conceptual capabilities that students do not possess.

It is also possible that the students' behavior within the institutional culture is biased towards a particular method in teaching mathematics, in general, and inequalities in particular. At the same time, students have prioritized the combined use of symbolic, verbal, numeric, but not pictorial representations while reasoning.

The study's findings suggest that an increased understanding of the solution of the polynomial inequalities by the graphical method has been reached, but that the integration of simple or elementary schemes into more comprehensive schemes or instrumented action schemes requires a high level of mastery of the component schemes, and that the instrumentation of CAS is a difficult process.

It was generally expected that students would instrument CAS to promote their reasoning and hence their understanding of the topic of polynomial inequalities, as a result of their interaction with CAS. Yet, students' work still showed weaknesses both in the use of the tools, their reasoning process and also in the understanding of polynomial inequalities.

4.1.2.Second research question:
What is the relation between students' CAS techniques and paper-and-pencil techniques when solving polynomial inequalities? What are the transfer and adaptation techniques between CAS environment and paper-and-pencil environment when solving polynomial inequalities?

In the P&P environment, students implemented numerical calculation techniques, while, in the CAS environment they implemented a number of elementary usage techniques including the graphing technique, a technique for determining x-intercepts and a technique for determining the coordinates of the extremum points. Later, students integrated some of these elementary techniques into composed or complex techniques; for example, the composed technique for investigating the graph behavior of polynomial functions, the composed technique for investigating the way the sign of polynomial functions varies (in the GV), and the complex technique for solving polynomial inequalities (in the GV).

On some occasions, the implementation of techniques acquired a heuristic value, as this was dependent on experience and routinization, while the techniques' epistemic values remained limited. This can be inferred because the output from the CAS did not elicit a need, among students, for the epistemic value to be derived from P&P techniques.

Moreover, it can be inferred that the use of CAS created the possibility of checking students' P&P solutions and clarified the methods in the two environments namely, the P&P and the CAS environments. For example, students used the graphical method to check the numerical calculations method while, on other occasions, they used the numerical calculations method and the table-of-signs method to check the solutions obtained by the graphical method. In rare cases, a transfer of techniques (from P&P to CAS) was identified, such as when students (groups HAG and MAG2) investigated, in the GV, the way the sign of the given functions varied or when they solved polynomial inequalities by graphical methods.

The study's findings suggest that the P&P and the CAS methods complement each other and improve the students' understanding of the topic. Yet, the easiness of implementing CAS techniques overshadowed the P&P techniques till the end of the instructional activities, except for some numerical calculation techniques (in P&P) that students implemented as their way for checking the validity of CAS solutions.

4.1.3. Third research question:
What difficulties (technical or conceptual) do students experience when using CAS (in the GeoGebra environment) to solve polynomial inequalities?

4.1.3.1. Difficulties:

The difficulties that students encountered are delineated herein:

4.1.3.1.1. Achievement level :

The available data suggests that low achieving students (group LAG2) acquired some technical knowledge related to the machine (CAS) and hence remained at the first extreme of Pierce and Stacey's continuum (2004). Other students (group HAG and MAG2) occupied an intermediary position of the continuum where "there is a substantial body of knowledge involving both mathematics and the machine" (Pierce & Stacey, 2004, p. 4).

Moreover, in this study, students' work method is identified as "a *calculator-restricted* work method". This, according to Guin and Trouche (2002), is characterized by "information sources more or less restricted to calculator investigations and simple manipulations" (p. 207).

4.1.3.1.2. Familiarity with CAS:

Students had no previous experience (before the unit about quadratic functions) with CAS, and using this type of technology was new to them. In addition, the experimental period was relatively short, which may have been a factor that resulted in a lack of overview about CAS and in hindering the mastering of the techniques involved. Students did not know about the availability of some of the tools that were necessary to solve the tasks at hand, and hence much of the instrumentation process of CAS remained unveiled. For example, to determine the coordinates of the extremum point of graphs, students did not know about the availability of the *Minimum* tool and *Maximum* tool (from the *Input Bar*); however, they were able to overcome this difficulty by using other tools to determine these coordinates. For example, they used the *New Point* tool (from the *Construction* menu).

4.1.3.1.3. Language:

Students (groups HAG, MAG2 and LAG2) struggled with sorting out the meaning of words such as explain, describe, deduce, pattern, graph behavior, sense of variations, and so on; therefore, impeding algebraic formalism. Language deficiency complicated students' description of the different solutions.

5. Discussion:

The study reveals that computer algebra does indeed offer opportunities for students to promote their understanding, to a limited extent, about methods for solving polynomial inequalities by graphical methods. Students mostly used the CAS for exploration and for implementing previously acquired solution strategies and expressed solutions of the different activities by using their natural mathematical language while using interval notation and CAS notation forms.

Despite the fact that the sample of participants (one class) was limited, yet some of the study's findings might go beyond the topic of polynomial inequalities. For example, computer algebra use contributed to the topic of "transformations" which, being important for all grade levels, presents teachers and students with many difficulties and is not treated by the Lebanese curriculum in a consistent manner across grade levels. Here, it can be concluded that the generalizability of the findings to other grades and levels, suggests even larger instrumentation difficulties in grades lower than the ninth. For higher grades, similar difficulties can be expected as those reported in this research; however, students will have more mathematical experience with which to overcome them. Students, mostly, used the CAS for exploration and for implementing previously acquired solution strategies and expressed solutions of the different activities by using their natural mathematical language while using interval notation and CAS notation forms.

The results of the research reveal that the instrumentation of CAS was limited and that students encountered several technical and conceptual difficulties. For example, students' knowledge of the availability of some tools and commands, in addition to syntactic difficulties persisted (though not with the same intensity) throughout the implementation of the instructional activities. Other difficulties included language difficulties and difficulties with linking techniques in the two environments P&P and CAS. Conceptual difficulties encompassed linking the different representations (tabular, graphical and algebraic) of functions. It is worth mentioning that most of these difficulties were not foreseen. An outcome of the research that was not expected beforehand concerns the importance of language, whole class discussions and the students' familiarity with CAS.

The research findings show, in line with Doorman, Drijvers, Gravemeijer, Boon, and Reed (2012), that CAS supported students' explorative activities for investigating the graph behavior of polynomial functions and, later, for investigating the way the sign of these functions varied. According to this study, CAS plays a limited epistemic mediation role. Again technical and conceptual difficulties prove, according to the study's findings and in line with those of Drijvers and van Herwaarden (2001), the existence of difficulties that hinder the instrumentation process.

The availability of multiple environments (algebraic, graphics and CAS) offer students opportunities to combine different representations, and hence, to promote their reasoning process. Yet, students mostly used graphical reasoning and considered numerical reasoning as a primary reasoning method to check their results. CAS, according to the current study's findings and in line with those of Doorman et al. (2012), helps students to shift from reasoning that depends on calculations with discrete integers to the graphical reasoning that depends on intervals of real numbers.

At the same time, the results of this research indicate that CAS allows students to check their P&P solutions, of the instructional activities, in ways that would have been very hard using P&P alone. Also, students sometimes use numerical calculations in P&P to check solutions that they got by using CAS. Consequently, the study's findings highlight the complementary roles of the techniques in the two environments, namely, the P&P and CAS environments. These findings are similar to what Kieran and Drijvers (2006) and Davis and Fonger (2015) suggested the complementary role of P&P and CAS techniques.

The research's findings do not reveal that all students reached a structural understanding of polynomial functions within the timeline of the teaching sequence. In line with Drijvers' (2003) and Doorman et al.'s findings (2012), this research's findings conclude that integrating CAS into algebra education is "better suited for longer periods, so that students would be able to really get used to the tool, or in higher grades, when they have more algebraic experience"(p. 296), and where the instrumentation difficulties are less dominant. Allowing more time for CAS training and for the implementation of the instructional activities would have led to a better instrumentation of CAS, is yet another result of this study. The achieved results agree with the results of a study by Trouche (2005 b) who indicates that time is necessary for analyzing the process of instrumentation in students' activity. In addition, the results of these aforementioned studies agree with the current research in what relates the reciprocal relation between technical and conceptual understanding.

The importance of students' achievement level and their familiarity with CAS, on the development and value of techniques and usage schemes, is apparent as some students (high and low achievers) were able to build a number of these schemes and techniques, while others (low achievers) were not able to build similar schemes and techniques. This result is supported by several researchers

(Guin and Trouche,1999; Artigue, 2002; and Kieran, 2007).

Teacher's inadequate and inefficient orchestration of whole class discussions, with other factors, have led to a limited instrumentation of CAS. This is in line with the results of studies by Drijvers and van Herwaarden (2001), Drijvers (2003), Artigue (2002) and Trouche (2004), which indicate the important role of the teacher in the instrumentation process.

6. Recommendations :

The results of this research verify that the use of CAS does improve problem-solving skills and contributes to a better understanding of mathematical concepts. Additionally, students acquire gains in problem-solving abilities and engagement with mathematics. In fact, such issues were similarly concluded in a recent book titled "Teaching Secondary Mathematics" (Hine, Reaburn, Anderson, Galligan, Carmichael, Cavanagh, Ngu and White, 2016, p. 108).

The loss of skill can be used as a reason for not adopting CAS systems; however, the current study has shown that this is not the case and students' skills did improve. Consequently, it is recommended to introduce CAS systems in the 10^{th} and 11^{th} grades during the full academic years. Further, and surprisingly, students tend to use CAS tools much more often than their teachers. Actually, "students use CAS tools to manipulate variables and look at their outcomes very quickly" (ibid). According to this observation, and in order to achieve the full benefits of the use of CAS systems, all teachers must be trained during the summer time and before starting the academic year on the full potentials of the system. Such training would allow the students to be exposed to this experience under the full supervision and support of their teachers. The aforementioned recommendation is supported by Kahn and Kyle (2002) who contend that "approximately 50% of the institutions conducting studies on the impact of technology reported increases in conceptual understanding, greater facility with visualization and graphical understanding, and an ability to solve a wider variety of problems…" (p. 59).

As for *teaching algebra using computer algebra*, the results suggest that it is important to orchestrate individual and collective instrumentation, to have students compare CAS techniques with P&P techniques, and to have students reflect on the way CAS works.

In terms of *further research* on learning mathematics in a technological environment, the researchers recommend that the scope of this research be extended to other mathematical concepts, such as the concept of function after participants have undergone an extensive training in CAS and its tools before shifting to the math topics.

7. The significance of the study:

This study highlights the issues of CAS integration into the Lebanese curriculum and contributes to filling the gap in research within Lebanese (and maybe Arab) contexts. Moreover, the study highlights the matter of organization of the topics within the Lebanese curriculum, taking the topics of *inequalities* and *functions* as an example.

The results provide a vision for schools, for administrators and professional development teams, about using technology in general, and CAS in particular, for teaching and learning of mathematics across the different grade levels. Likewise, the study findings offer feedback about the barriers that might hinder CAS integration. Once these barriers are identified, effective integration plans can be developed. The results of the study foster a better understanding, by teachers, of the potentials of CAS in algebra classes and how the use of this technology can benefit their teaching and learning practices. Moreover, the results help in alleviating some of the problems that arise when students work in CAS and in a paper-and-pencil environment.

8. Limitations of the study:

The results would have been more revealing if the number of CAS-based sessions and the time allotted to the study had been more. However, students generally lacked the familiarity with CAS.

Whole-class demonstrations and discussions should have received more attention during the teaching experiments. The group's interactions and discussions and mediation were not steered productively in a manner that encourage the production of appropriate usage schemes. The teacher's stimulated interventions during the whole class discussions were not adequately managed and consequently, the advances made by the students were not remarkable. Moreover, it is worth mentioning that this research was carried out with a limited sample (in only one class) which limits the generalization of the results.

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Exhibit 1. Pretest

Appendix A: Pretest

Group#	**Name**	**Activity**	**Date**
		Pretest	

The aim of this test is to have an idea of your previous experience with solving linear inequalities graphically and by algebraic calculations in a paper-and-pencil environment and without using CAS.

Instructions:

- *Don't forget to write the necessary indicators: the* ***number of the group and your name.***
- *Don't erase or overwrite any part of your paperwork. If necessary, put the part that you want to omit, within brackets and indicate that it is not needed.*
- *Use only the provided paper for writing all your solutions.*
- *In case you need draft paper, use the back of your solution sheet after writing the number of the question and the corresponding part.*
- *Please note that no questions are to be left unanswered*

Question 1
Objectives:
Solve linear inequalities by algebraic calculation and represent the solutions on a number line.
Duration: One period (50 minutes)
The context of the question:

1. Which values of x, given in the table below, are a solution of the given inequality? Justify?

	$x =$	-100	-3	-1.4	0	6
$3(x-2) \geq -2x - 13$						

2. Solve the following inequalities and represent your solution using a number line:
   a. $x - 45 \leq 13$
   b. $3x - 18 \geq 5x + 21$

Question 2
Objectives:
Plot the graphs of straight lines
Solve linear inequalities graphically and justify the solutions verbally.

Duration: One period (50 minutes)
The context of the question:
In the **same** orthonormal system (x'ox, y'oy):

1. Plot the graph of the function f such as $f(x) = 2x - 5$.
2. Plot the graph of the function g such as $g(x) = -x + 4$
3. Complete the statements below about the suitable values of x, and justify:
   a. $2x - 5 \leq 0$ when x is …………………
   b. $-x + 4 > 0$ when x is …………………
   c. $2x - 5 > 3$ when x is …………………
   d. $-x + 4 < 4$ when x is …………………
   e. $-x + 4 \geq 2x - 5$ when x is …………………

Exhibit 2.a. Outline of the first instructional activity

Group#	Session number	Activity	Date
	S1	Act 1	

Activity one (Act1)

- *Steps for software use*

Note: For each step, you read start applying directly.

- *Replace each **# symbol** by the required number.*
- *Make sure that your folder, for storing GeoGebra files is ready under the name **GG.***
- *Make sure that your folder, for storing DVC files, is ready under the name **DVC***
- *Open DVC and start recording with Debut Video Capture (DVC) before you start working with GeoGebra.*
- *Open GeoGebra.*
- *Start working on your activity.*
- *Every **<u>five minutes:</u>***

1. *Stop working.*
2. *Save your GeoGebra file on the desktop folder **GG** under the name **G#S1Act1GG#** to indicate the **number of the group, then the number of the session**, followed by the **number of the activity and the number of the GeoGebra file.***
3. *Stop recording and save your recorded DVC file in the folder **DVC** under the name **G#S1Act1Vid#** to indicate the **number of the group, then the number of the session and activity**, followed by the **number of the video.***
4. *Resume working on your activity after operating DVC and opening a new GG window.*
5. *Repeat the above steps every five minutes until you finish the activity.*

- *When you finish working on your activity, copy folders **GG and DVC** to the provided CD and hand in the CD to your teacher.*
- *Don't ever try to delete or modify or edit any file or part of the files on the laptop.*

- ***Steps for paper use***
  - *At the top of this file, don't forget to write your group's name in the required cell.*
  - *Don't erase or overwrite any part of your paperwork. If necessary, put the part that you want to omit within brackets and indicate that it is not needed.*
  - *If there is more than one way for solving apart, don't hesitate to write them.*
  - *If you need additional space to solve any part, use the back of the pages after writing the number of the part*

Exhibit 2.b. Activity 1

- **Activity 1:**

1. Complete the table below by finding the value of the given expression $f(x)$ for the indicated values of x:

When $x =$	-3	-2	2	3	6
The value of the expression $f(x) = (3x-1)+4(x-5)$					

2. In this part, it is required to **explain how the sign of $f(x)$ would vary if you include other values of x to the above table**.

Complete the statements below, by filling the blank space with the correct number, based on your observations of the results in the above table

a. In the table above, the expression $f(x) = (3x-1)+4(x-5)$ will be ……… (**negative/positive**) if we add other values of x which are less than ………

b. In the table above, the expression $f(x) = (3x-1)+4(x-5)$ will be ……… (**negative/positive**) if we add other values of x which are greater than ……..

3. In this part, it is required to **explain how you would use GeoGebra to determine the sign of the expression $f(x)$**.

a. **Explain** how you would use GeoGebra to determine the sign of the expression: $f(x) = (3x-1)+4(x-5)$. If more than one way is possible, explain these ways.

b. **Explain** how you would **deduce** the sign of the expression: $f(x) = (3x-1)+4(x-5)$ when $x = +4$.

c. **Explain** how you would **deduce** the sign of the expression: $f(x) = (3x-1)+4(x-5)$ when $x = -6$

Exhibit 3.a. Outline of the second instructional activity

Group	Session number	Activity	Date
	S2	Act 2	

Activity two (Act2)

- **Steps for software use**

Note: For each step, you read start applying directly.

- Replace each # **symbol** by the required number.
- Make sure that your folder, for storing GeoGebra files, is ready under the name ***G#S#Act#GG.***
- Make sure that your folder, for storing DVC files, is ready under the name ***G#S#Act#DVC***
- Open DVC and start recording with Debut Video Capture (DVC) before you start working with GeoGebra.
- Open GeoGebra.
- Start working on your activity.
- Every **five minutes:**

1. Stop working.
2. Save your GeoGebra file on the desktop folder ***GG*** under the name ***G#S2Act2GG#*** to indicate the ***number of the group, then the number of the session***, followed by the ***number of the activity and the number of the GeoGebra file.***
3. Stop recording and save your recorded DVC file in the folder ***DVC*** under the name ***G#S2Act2Vid#*** to indicate the ***number of the group, then the number of the session and activity***, followed by the ***number of the video.***
4. Resume working on your activity after operating DVC and opening a new GG window.
5. Repeat the above steps every five minutes until you finish the activity.

- When you finish working on your activity, copy folders ***GG and DVC*** to the provided CD and hand in the CD to your teacher.
- Don't ever try to delete or modify or edit any file or part of the files on the laptop.

- **Steps for paper use**

- At the top of this file, don't forget to write your group's name in the required cell.
- Don't erase or overwrite any part of your paperwork. If necessary, put the part that you want to omit within brackets and indicate that it is not needed.
- If there is more than one way for solving apart, don't hesitate to write those ways.
- If you need additional space to solve any part, use the back of the pages after writing the number of the part

Exhibit 3.b. Activity 2

- **Activity 2:**
  1. In this part, it is required to **describe the sense of variation of *even-powered* polynomial functions** that are represented by the graphs that you see in **GeoGebra**.
  - Use GeoGebra to plot the graphs of the *even powered* polynomial functions:
    $f(x) = x^2 + 2$, $g(x) = x^4 - 1$, and $h(x) = x^6 - 2x^5$ and $p(x) = 2x^8 - \frac{1}{2}x$.
  - **Describe** the pattern that you notice about the **graph behavior** (**sense of variation**) of the even-powered polynomial functions that you see. Specify if the graph admits a minimum or a maximum (**without calculating** them) then indicate the **number** of points of intersection with the x-axis.

  2. **Deduce** the sense of variation of the *even-powered* polynomial function $K(x)$ where $K(x) = 5(3x-1)(x+4)$. **Explain** and **justify** your answer.

  3. **Explain** how you would use **GeoGebra** and/or **paper-and-pencil** to determine the sign of $K(x)$ for the values of x given in the table below. More than one method is possible.

$x =$	-5	-2	0	9
Sign of $K(x) = 5(3x-1)(x+4)$				

  4. Explain how you would use **GeoGebra and/or paper-and-pencil** to study the sign of **any even powered polynomial** function. More than one method is possible

  5. Use **GeoGebra** and/or **paper-and-pencil** to solve the inequality below:
     $4x^4 + 4x^3 - 33x^2 - 9x + 54 < 0$.
     More than one method is possible. Explain your work in details

Exhibit 4.a. Outline of the third instructional activity

Group	Session number	Activity	Date
	S3	Act 3	Tuesday, May 24, 2016

Activity three (Act3)

- **Steps for software use**
  Note: For each step, you read start applying directly.

- Replace each **# symbol** by the required number.
- Make sure that your folder, for storing GeoGebra files, is ready under the name ***G#S3Act3GG.***
- Make sure that your folder, for storing DVC files, is ready under the name ***G#S3Act3DVC***
- Open DVC and start recording with Debut Video Capture (DVC) before you start working with GeoGebra.
- Open GeoGebra.
- Start working on your activity.
- Every **five minutes**,
  1. Stop working.
  2. Save your GeoGebra file on the desktop folder ***G#S3Act3GG*** under the name ***G#S3Act3GG#*** to indicate the ***number of the group, then the number of the session***, followed by the ***number of the activity and the number of the GeoGebra file.***
  3. Stop recording and save your recorded DVC file in the folder ***G#S3Act3DVC*** under the name ***G#S3Act3Vid#*** to indicate the ***number of the group, then the number of the session and activity***, followed by the ***number of the video.***
  4. Resume working with your activity after opening operating DVC and opening a new GG window.
  5. Repeat the above steps every five minutes until you finish the activity.
- When you finish working with your activity, copy folders ***GG and DVC*** to the provided CD and handle the CD to your teacher.
- Don't ever try to delete or modify or edit any file or part of the files on the laptop.

- **Steps for paper use**

- At the top of this file, don't forget to write your group's name in the required cell.
- Don't erase or overwrite any part of your paperwork. If necessary, put the part that you want to omit within brackets and indicate that it is not needed.
- If there is more than one way for solving apart, don't hesitate to write those ways.
- If you need additional space to solve any part, use the back of the pages after writing the number of the part

Exhibit 4.b. Activity 3

- **Activity 3**
1. In this part, it is required to **describe the sense of variation of *odd-powered* polynomial functions** that are represented by the graphs that you see in **GeoGebra**.
- Use GeoGebra, to plot the graphs of the *odd powered* polynomial functions:
  $f(x) = x^3 - 4x$, $g(x) = x^5$, $h(x) = x^7 + 2x - 1$ and of $p(x) = x^9 - 3x^2 - \frac{1}{2}$.
- **Describe** the pattern that you notice about the graph behavior (sense of variation) of the odd - powered polynomial functions that you see. Specify if the graph admits a minimum or a maximum (without calculating them) then indicate the number of points of intersection with the x-axis.

2. **Deduce** the sense of variation of the *even-powered* polynomial function $T(x)$ where $T(x) = (3x - 9)(x^2 + 2x - 3)$. **Explain** and **justify** your answer.

3. **Explain** how you would use **GeoGebra** and/or **paper-and-pencil** to determine the sign of $\mathrm{T}(x)$ for the values of x given in the table below:

	$x = -10$	$x = -1$	$x = 3$	$x = 17.5$
Sign of T(x)=$(3x - 9)(x^2 + 2x - 3)$	?	?	?	?

4. **Explain** how you would you would use **GeoGebra** and/or **paper-and-pencil** to study the sign of any odd-powered polynomial function.

5. Use **GeoGebra** and/or **paper-and-pencil** to solve the following inequality below: $x^5 - 3x^4 - 8x^2 + 24x \geq 0$. Explain your work in details

Exhibit 5.a. Outline of the fourth instructional activity

Group	**Session number**	**Activity**	**Date**
	S4	Act 4	

Activity four (Act4)

- **Steps for software use**

Note: For each step, you read start applying directly.

- Replace each **# symbol** by the required number.
- Make sure that your folder, for storing GeoGebra files, is ready under the name ***G#S4Act4GG.***
- Make sure that your folder, for storing DVC files, is ready under the name ***G#S4Act4DVC***
- Open DVC and start recording with Debut Video Capture (DVC) before you start working with GeoGebra.
- Open GeoGebra.
- Start working on your activity.
- Every **five minutes**,
  1. Stop working.
  2. Save your GeoGebra file on the desktop folder ***G#S4Act4GG*** under the name ***G#S4Act4GG#*** to indicate the ***number of the group, then the number of the session***, followed by the ***number of the activity and the number of the GeoGebra file.***
  3. Stop recording and save your recorded DVC file in the folder ***G#S4Act4DVC*** under the name ***G#S4Act4Vid#*** to indicate the ***number of the group, then the number of the session and activity***, followed by the ***number of the video.***
  4. Resume working on your activity after operating DVC and opening a new GG window.
  5. Repeat the above steps every five minutes until you finish the activity.
- When you finish working on your activity, copy folders ***G#S4Act4GG and G#S4Act4DVC*** to the provided CD and handle the CD to your teacher.
- Don't ever try to delete or modify or edit any file or part of the files on the laptop.

- **Steps for paper use**

- At the top of this file, don't forget to write your group's name in the required cell.
- Don't erase or overwrite any part of your paperwork. If necessary, put the part that you want to omit within brackets and indicate that it is not needed.
- If there is more than one way for solving apart, don't hesitate to write those ways.
- If you need additional space to solve any part, use the back of the pages after writing the number of the part

Exhibit 5.b. Activity 4

Activity 4 (CAS technique activity)

1. Use **GeoGebra** to solve the inequalities below. Explain your work in details.
   - $(x-1)(2x-3)(3x-2)^3(x-2)^5 > 0$
   - $(5x-1)(x+2)^3(4x+5)^3 \leq 0$
2. Explain how you would use **paper-and-pencil** to solve the above inequalities. Explain your steps in details.