Spring 2026 − Math 3331−Variation of Parameters

The solution of linear nonhomgeneous ODEs

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

is given by

$$y = y_h + y_p. \tag{2}$$

The y_h is the solution of the homogeneous problem equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

and y_p some solution of the entire problem (called the particular solution).

Variation of Parameters

This is a variation of reduction of order. Given two solutions of a second order homogeneous ODE (say y_1 and y_2) we seek the particular solution using

$$y_p = u(x)y_1 + v(x)y_2. \tag{4}$$

As we have two unknowns (u and v), we will need two equations. Differentiating (4) we obtain

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'. \tag{5}$$

We first set

$$u'y_1 + v'y_2 = 0 \ \ \text{(our first equation)}. \tag{6}$$

This gives (5) as

$$y_p' = uy_1' + vy_2'. \tag{7}$$

we substitute this and one more derivative to the original ODE (1) giving

$$u'y_1' + v'y_2' = f(x) \ \ \text{(our second equation)}. \tag{8}$$

We then solve (6) and (8) for u' and v' and integrate each. We then substitute into (4) to obtain the particualr solution.

The follow examples illustrate.

Example 1.

Solve

$$y'' - 3y' + 2y = 1 + e^x \tag{9}$$

We first consider the homogeneous equation

$$y'' - 3y' + 2y = 0 \tag{10}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0 \tag{11}$$

which has roots $r = 1, 2$ so two independent solutions are $y = e^x$ and $y = e^{2x}$. We seek a particualr solution of the form

$$y = e^x u + e^{2x} v \tag{12}$$

We calculate a derivative

$$y' = e^x u' + e^x u + e^{2x} v' + 2e^{2x} v \tag{13}$$

Set set

$$e^x u' + e^{2x} v' = 0 \tag{14}$$

so

$$y' = e^x u + 2e^{2x} v \tag{15}$$

We calculate another derivative

$$y'' = e^x u' + e^x u + 2e^{2x} v' + 4e^{2x} v \tag{16}$$

Now substitute (15) and (16) into (9) giving

$$\begin{aligned} &y'' - 3y' + 2y = 1 + e^x \\ &e^x u' + e^x u + 2e^{2x} v' + 4e^{2x} v - 3\left(e^x u + 2e^{2x} v\right) + 2\left(e^x u + e^{2x} v\right) = 1 + e^x \\ &e^x u' + 2e^{2x} v' = 1 + e^x \end{aligned} \tag{17}$$

So we simplify

$$\begin{aligned} e^x u' + e^{2x} v' &= 0 \\ e^x u' + 2e^{2x} v' &= 1 + e^x, \end{aligned} \tag{18}$$

two equations for u' and v'. Solving for each gives

$$e^x u' = -1 - e^x, \quad e^{2x} v' = 1 + e^x \tag{19}$$

or

$$u' = -e^{-x} - 1, \quad v' = e^{-2x} + e^{-x} \tag{20}$$

Integrating and neglects constant of integration gives

$$u = e^{-x} - x, \quad v' = -\frac{1}{2}e^{-2x} - e^{-x} \tag{21}$$

From (5)

$$\begin{aligned} y_p &= uy_1 + vy_2 \\ &= \left(e^{-x} - x\right)e^x + \left(-\frac{1}{2}e^{-2x} - e^{-x}\right)e^{2x} \\ &= 1 - xe^x - \frac{1}{2} - e^x \\ &= \frac{1}{2} - xe^x - e^x \end{aligned} \tag{22}$$

so the general solutions is

$$y = y_h + y_p = c_1e^x + c_2e^{2x} + \frac{1}{2} - xe^x - e^x \tag{23}$$

noting the last term e^x can be absorbed into the first term c_1e^x.

Example 2.

Solve

$$y'' + y = \tan x \tag{24}$$

We first consider the homogeneous equation

$$y'' + y = 0 \tag{25}$$

The characteristic equation is

$$r^2 + 1 = 0 \tag{26}$$

which has roots $r = -i, i$ so two independent solutions are $y = \cos x$ and $y = \sin x$. We seek the homogeneous solution as

$$y = u\cos x + v\sin x \tag{27}$$

We calculate derivatives

$$y' = u'\cos x - u\sin x + v'\sin x + u\cos x \tag{28}$$

and set

$$u' \cos x + v' \sin x = 0 \tag{29}$$

so

$$y' = -u \sin x + v \cos x \tag{30}$$

Taking another derivative we obtain

$$y'' = -u' \sin x - u \cos x + v' \cos x - v \sin x \tag{31}$$

Substituting into (24) gives

$$-u' \sin x + v' \cos x = \tan x \tag{32}$$

We solve (29) and (32) for u' and v' giving

$$u' = -\frac{\sin^2 x}{\cos x}, \quad v' = \sin x \tag{33}$$

integrating each gives

$$u = \sin x - \ln|\sec x + \tan x|, \quad v = -\, cosx \tag{34}$$

so

$$\begin{aligned} y_p &= (\sin x - \ln|\sec x + \tan x|) \cos x + (-\cos x) \sin x \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned} \tag{35}$$

and the general solution

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x| \tag{36}$$

which we saw earlier.

Formulas

We obtain a particular solution of the nonhomogeneous ODE

$$y'' + p(x)y' + q(x)y = f(x), \tag{37}$$

using the variation of parameters form

$$y = uy_1 + vy_2 \tag{38}$$

where u and v satisfy

$$u = -\int \frac{y_2 f(x)}{W}\, dx, \quad v = \int \frac{y_1 f(x)}{W}\, dx \tag{39}$$

where y_1 and y_2 are solutions of the homogeneous problem and W, the Wronskian, defined by $W = y_1 y_2' - y_2 y_1'$.