Math 4315 PDE's

Method of Characteristics

Given $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$

we solve $\dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c}$

we pick in pairs to obtain

$$I_1(x,y,u) = c_1, \quad I_2(x,y,u) = c_2$$

the sol^n of the PDE is $c_2 = f(c_1)$.

The following examples illustrate

Ex 1 $u_x - 3u_y = 2x$

$$\frac{dx}{1} = \frac{dy}{-3} = \frac{du}{2x}$$

1st $3\,dx = -dy \Rightarrow 3x + y = c_1$

2nd $2x\,dx = du \Rightarrow x^2 = u - c_2 \Rightarrow c_2 = u - x^2$

sol^n $u - x^2 = f(3x+y)$ or $u = x^2 + f(3x+y)$

Ex 2 $\quad u_x + (x-y)\,u_y = 3$

$$\frac{dx}{1} = \frac{dy}{x-y} = \frac{du}{3}$$

1st pair $\quad \dfrac{dy}{dx} = x - y \quad$ a $\quad \dfrac{dy}{dx} + y = x \quad$ Linear

$$\mu = e^{+x} \qquad \frac{d}{dx}\, e^x y = x e^x$$

$$e^x y = (x-1)e^x + c_1 \quad \text{so} \quad c_1 = e^x(y - x - 1)$$

2nd pair $\quad 3\,dx = du$

$$3x = u - c_2 \quad \Rightarrow \quad c_2 = u - 3x$$

sol^n $\quad u - 3x = f\left(e^x(y-x-1)\right)$

a $\quad u = 3x + f\left(e^x(y-x-1)\right)$

Ex 3 $\quad$ From the Homework (later

$$u_t + u\,u_x = 0 \qquad u(x,0) = e^{-x^2}$$

CE. $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$ ↙ what's this means?

it means that $du = 0$ directly

so $u = c_1$

$dt = \frac{dx}{c_1} \Rightarrow c_1 dt = dx$

$\Rightarrow c_1 t = x - c_2$ so $c_2 = x - c_1 t$ ↙ bring back in c_1

$= x - tu$

sol^n $c_1 = f(c_2)$

$\Rightarrow u = f(x - tu)$

I.C. $u(x,0) = e^{-x^2} \Rightarrow e^{-x^2} = f(x)$

so $u = e^{-(x-tu)^2}$ sol^n - implicit

Ex 4 $y u_x + (x - u) u_y = y$

CE $\frac{dx}{y} = \frac{dy}{x-u} = \frac{du}{y}$ From HW 1.

1st pair

$$\frac{dx}{y} = \frac{du}{y} \Rightarrow \quad x = u + c_1 \qquad c_1 = x - u$$

2nd pair $\frac{dx}{y} = \frac{dy}{x-u} \Rightarrow \frac{dx}{y} = \frac{dy}{c_1}$

$$\Rightarrow \quad c_1\,dx = y\,dy$$

$$\Rightarrow \quad c_1 x = \frac{y^2}{2} + c_2 \quad \text{so} \quad c_2 = c_1 x - y^2/2$$

$$= x(x-u) - y^2/2$$

sol^n $c_2 = f(c_1)$

$$\Rightarrow \quad x(x-u) - \frac{y^2}{2} = f(x-u)$$ implicit

If we ~~had~~ the BC. $u(x,x) = 1$

then $x(x-1) - x^2/2 = f(x-1)$ let $\lambda = x-1$ so $x = \lambda + 1$

$$f(\lambda) = (\lambda+1)\lambda - \frac{(\lambda+1)^2}{2} = \frac{\lambda^2 - 1}{2}$$

so $x(x-u) - \frac{y^2}{2} = \frac{(x-u)^2 - 1}{2}$ or $u^2 = x^2 - y^2 + 1$