Math 4381/6378 Symmetry

So far we ~~have~~ talked about symmetries of 1^{st} order ODE

$$y' = F(x,y)$$

invariant under some Lie Group

$$\bar{x} = f(x,y,\varepsilon) \quad \bar{y} = g(x,y,\varepsilon)$$

However, it is the infinitesimal X & Y what we've really interested in

Why? If we solve

$$X r_x + Y r_y = 0 \quad X s_x + Y s_y = 1$$

then a change of variables

$$(x,y) \to (r,s)$$

leads to a separable ODE with no s_y

we note that X & Y satisfy

$$Y_x + (Y_y - X_x)F - X_y F^2 = X F_x + Y F_y$$

known as Lie's invariance condition

we now would like to extend our results to higher order ODE's

consider the ODE

$$y'' = x y'^2 + \frac{y^2}{x}$$

Show this ODE is invariant under

$$\bar{x} = e^{\varepsilon} x, \quad \bar{y} = e^{-\varepsilon} y$$

so

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} = \frac{e^{-\varepsilon} y'}{e^{\varepsilon}} = e^{-2\varepsilon} y'$$

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{\frac{d}{dx}\bar{y}'}{\frac{d\bar{x}}{dx}} = \frac{e^{-2\varepsilon} y''}{e^{\varepsilon}} = e^{-3\varepsilon} y''$$

so $\bar{y}'' = \bar{x}\bar{y}'^2 + \frac{\bar{y}^2}{\bar{x}}$

$$\Rightarrow e^{-3\varepsilon} y'' = x e^{\varepsilon} (e^{-2\varepsilon} y')^2 + \frac{(e^{-\varepsilon} y)^2}{e^{\varepsilon} x}$$

$$\cancel{e^{-3\varepsilon}} y'' = \cancel{e^{-3\varepsilon}} x y'^2 + \cancel{e^{-3\varepsilon}} \frac{y^2}{x}$$

$$y'' = x y'^2 + \frac{y^2}{x}$$ so yes invariant

Now let's continue as if the ODE were 1st ord

$$X = \frac{\partial \hat{x}}{\partial \varepsilon}\Big|_{\varepsilon=0} = e^{\varepsilon} x\Big|_{\varepsilon=0} = x$$

$$Y = \frac{\partial \hat{y}}{\partial \varepsilon}\Big|_{\varepsilon=0} = -e^{-\varepsilon} y\Big|_{\varepsilon=0} = -y$$

so we solve

$x r_x - y r_y = 0 \qquad x S_x - y S_y = 1$

so $r = R(xy)$ $s = \ln x + S(xy)$

d pick $r = xy$ $s = \ln x$

$\Rightarrow$ $x = e^{s}$ $y = r e^{-s}$

so $$\frac{dy}{dx} = \frac{e^{-s} - r e^{-s} s'}{e^{s} s'} = e^{-2s}\left(\frac{1-rs'}{s'}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dr} y'}{\frac{dx}{dr}} = \frac{\frac{d}{dr}\left(e^{-2s}\left(\frac{1-rs'}{s'}\right)\right)}{e^{s} s'}$$

$$= e^{-3s}\left(\frac{2rs'^3 - 3s'^2 - s''}{s'^3}\right)$$ maple helped

so sub into ODE

$$e^{-3s}\left(\frac{2rs'^3 - 3s'^2 - s''}{s'^3}\right) = e^{s}\cdot\left[e^{-2s}\left(\frac{1-rs'}{s'}\right)\right]^2 + \frac{r^2 e^{-2s}}{e^{s}}$$

a $s'' + (2r^2 - 2r)s'^3 + (3-2r)s'^2 + s' = 0$ no s if we let $s' = u$ then

$$u' + (2r^2 - 2r)u^3 + (3-2r)u^2 + u = 0 \quad 1^{st} \text{ order.}$$

so if we are lucky to solve ↑ then we have reduced the 2^{nd} order problem to 1^{st} order.

So how does we get X & Y?

1^{st} order

$$\bar{x} = x + \varepsilon X + O(\varepsilon^2)$$

$$\bar{y} = y + \varepsilon Y + O(\varepsilon^2)$$

$$\bar{y}' = y' + \varepsilon Y_{[x]} + O(\varepsilon^2)$$

then invariance

$$\bar{y}' = F(\bar{x}, \bar{y}) \quad \text{when} \quad y' = F(x, y)$$

$$y' + \varepsilon Y_{(x)} = F(x + \varepsilon X, y + \varepsilon Y) \quad \text{expand}$$

$$y' + \varepsilon Y(x) + O(\varepsilon^2) = F(x,y) + \varepsilon(XF_x + YF_y) + O(\varepsilon^2)$$

and if we know $y' = F(x,y)$

then to order $O(\varepsilon^2)$ we require

$$Y[x] = XF_x + YF_y \quad \text{(Lie's Invariance condition)}$$

when $y' = F$

2nd order

$$\left.\begin{aligned} \bar{x} &= x + \varepsilon X + O(\varepsilon^2) \\ \bar{y} &= y + \varepsilon Y + O(\varepsilon^2) \\ \bar{y}' &= y' + \varepsilon Y[x] + O(\varepsilon^2) \end{aligned}\right\} (*)$$

$$\bar{y}'' = y'' + \varepsilon Y[xx] + O(\varepsilon^2)$$

$$\bar{y}'' = F(\bar{x}, \bar{y}, \bar{y}') \quad \text{when} \quad y'' = F(x, y, y')$$

sub $(*)$ & expand so to $O(\varepsilon^2)$

$$Y[xx] = XF_x + YF_y + Y[x]F_{y'} \quad \text{when } y'' = F$$

we know what $Y[x]$ is so now we need to know $Y[xx]$.

Recall if

$$\bar{x} = x + \varepsilon X(x,y) + O(\varepsilon^2)$$
$$\bar{y} = y + \varepsilon Y(x,y) + O(\varepsilon^2)$$

then
$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx}\left(y + \varepsilon Y(x,y) + O(\varepsilon^2)\right)}{\frac{d}{dx}\left(x + \varepsilon X(x,y) + O(\varepsilon^2)\right)}$$

Let us define D_x (total differential operator)

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'}$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{y' + \varepsilon D_x(Y) + O(\varepsilon^2)}{1 + \varepsilon D_x(X) + O(\varepsilon^2)}$$

$$= \left(y' + \varepsilon D_x(Y)\right)\left(1 - \varepsilon D_x(X)\right) + O(\varepsilon^2)$$

$$= y' + \varepsilon\left[D_x(Y) - y' D_x(X)\right] + O(\varepsilon^2)$$

let $Y[x] = D_x(Y) - y' D_x(X)$

so
$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{\frac{d}{dx}\left(\frac{d\bar{y}}{d\bar{x}}\right)}{\frac{d}{dx}\bar{x}} = \frac{\frac{d}{dx}\left(y' + \varepsilon Y[x] + O(\varepsilon^2)\right)}{1 + \varepsilon D_x(X)}$$

$$= \left(y'' + \varepsilon D_x(Y(x))\right)\left(1 - \varepsilon D_x(X)\right) + O(\varepsilon^2)$$

$$= y'' + \varepsilon\left(D_x(Y[x]) - y'' D_x(X)\right) + O(\varepsilon^2)$$

so define

$$Y[xx] = D_x(Y[x]) - y'' D_x(X)$$

so Lie's invariance condition

$$Y[xx] = X F_x + Y F_y + Y[x] F_{y'} \quad \text{when } y'' = F$$

Now lets expand $Y[xx]$

so $Y[xx] = D_x(Y[x]) - y'' D_x(X)$

$$= D_x\left(Y_x + Y_y y' - X_x y' - X_y y'^2\right) - y''(X_x + X_y y')$$

$$= Y_{xx} + Y_{xy} y' + Y_{xy} y' + Y_{yy} y'^2 + Y_y y''$$

$$- X_{xx} y' - X_{xy} y'^2 - X_x y'' - X_{xy} y'^2 - X_{yy} y'^3 - 2 X_y y' y''$$

$$- X_x y'' - X_y y' y''$$

$$= Y_{xx} + (2Y_{xy} - X_{xx}) y' + (Y_{yy} - 2X_{xy}) y'^2$$

$$- X_{yy} y'^3 + (Y_y - 2X_x) y'' - 3X_y y' y''$$

and Lie's invariance condition is

$$Y[xx] = X F_x + Y F_y + Y[x] F_{y'}$$

when $Y[x]$ is given previously, $Y[xx]$ above

and we set $y'' = F(x, y, y')$

ex 1 $y'' = 0$

so Lie's invariance condition becomes

$$Y_{xx} + (2Y_{xy} - X_{xx})y' + (Y_{yy} - 2X_{xy})y'^2 - X_{yy}y'^3 = 0$$

since X & Y are independent of y' means

$$\left.\begin{aligned} Y_{xx} &= 0 \\ 2Y_{xy} - X_{xx} &= 0 \\ Y_{yy} - 2X_{xy} &= 0 \\ X_{yy} &= 0 \end{aligned}\right\} \text{ Called "determining equations"}$$

Note: Subtle difference - there are 4 equations to determine the form of X & Y opposed to a single equation for $y' = F$.

Sol^n of DE's

$X_{yy} = 0 \Rightarrow X = A(x)y + B(x)$

$Y_{yy} - 2X_{xy} = 0 \Rightarrow Y_{yy} = 2A'(x)$

so $Y_y = 2A'(x)y + C(x)$

$Y = A'(x)y^2 + C(x)y + D(x)$

so $Y_{xx} = 0$ $\quad A'''(x)y^2 + C''(x)y + D''(x) = 0$

$\Rightarrow A''' = 0, \; C'' = 0 \quad D'' = 0$

$2Y_{xy} - X_{xx} = 0 \Rightarrow 4A''(x)y + 2C'(x) - A''(x)y - B''(x) = 0$

$\Rightarrow A'' = 0 \quad 2C'(x) - B'' = 0$

so $A = c_1 x + c_2, \; C = c_3 x + c_4 \quad D = c_5 x + c_6$

$B'' = 2c_3 \Rightarrow B = c_3 x^2 + c_7 x + c_8$

$X = (c_1 x + c_2)y + c_3 x^2 + c_7 x + c_8$

$Y = c_1 y^2 + (c_3 x + c_4)y + c_5 x + c_6$

8 parameter family of symmetries