Junior 2014 Solutions

1. Call a triangle simple if each of its sides has length $4, 6$, or 8. Find all different simple triangles.
Solution. There are a total of 10 possible triplets that forms distinct triangles. However, by the triangle inequality, only 9 of them form non-degenerate triangles.

$$(4,4,4)$$
$$(6,6,6)$$
$$(8,8,8)$$
$$(4,4,6)$$
$$(4,6,6)$$
$$(4,8,8)$$
$$(4,6,8)$$
$$(6,6,8)$$
$$(6,8,8)$$

2. In the following, there are 8 possible paths to spell the word MATH:

M
A A
T T T
H H H H

In each move you can only go up or down by one row. Determine the number of paths that spell the word LEVEL in the following

L
E E
V V V
E E
L

Solution. There are four orientations to view the word LEVEL. In each orientation there are 6 paths. Therefore, there are 24 paths. Each path follows the pattern of the Pascal's triangle. For example, starting from the top L and going to the bottom L gives

1
1 1
1 2 1
3 3
6

3. In the Fibonacci sequence, $1, 1, 2, 3, 5, \ldots$, each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

Solution. The sequence follows the pattern of odd, odd, even, odd, odd, even, odd, odd, even, ... In the first 100, terms, there are a total of 33 repetitions of odd, odd, even plus one more odd as the 100^{th} term. Therefore, there are $(33)(2) + 1 = 67$ odd terms.

4. Four people A, B, C, D are enrolled in some number of courses. A is taking 8 courses and B is taking 5 courses. It is known that A is taking more courses than anyone else and B is taking less courses than anyone else. It is also known that exactly three of these four people are enrolled in each math course. How many courses are offered?

Solution. Since the number of courses taken by C or D is greater than 5 and less than 8, then the number of courses taken by C and D is either $12, 13$, or 14. Therefore, the total number of courses taken by A, B, C, and D is $25, 26$, or 27. Since the sum of the number of people throughout all the courses is a multiple of 3, the total number of courses taken by the four people must be 27. Therefore, there are $\frac{27}{3} = 9$ courses.

5. Out of six children, exactly two were known to have been stealing apples. But who? Helen said "Christine and George". Jane said "Donald and Tom". Donald said "Tom and Christine". George said "Helen and Christine". Christine said "Donald and Jane". Tom couldn't be found. Four of the children actually named one of the culprits but lied about the other. The fifth child lied about both. Who stole the apples?

   Solution. First, we organize the give information.

Accuser	Accused
Christine	Donald, Jane
Donald	Christine, Tom
George	Christine, Helen
Helen	Christine, George
Jane	Donald, Tom

   Assume George is guilty. Since George is guilty, then Christine is not since no one tells the truth about both. Since Christine is not guilty, then exactly one of Tom or Helen must be guilty, otherwise both Donald and George have lied about both. Regardless of whether it is Tom or Helen that is guilty, either Donald or George lied about both. Christine has also lied about both. This is not possible since only one of the five children lied about both. Therefore, George is innocent.
   Helen is also named exactly once. If we assume that Helen is guilty, a similar argument leads to a contradiction. Therefore, Helen is innocent.
   If Christine was not guilty, then George and Helen lied about both. Since this is not possible, Christine is guilty.
   If Donald is guilty, then every child told the truth once. Since this is not possible, Donald is innocent. This leaves only Jane, who must be the second guilty child. Therefore, Christine and Jane stole the apples.

Senior 2014 Solutions

1. A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 1, a_2 = 4, \text{ and } a_n = 2a_{n-1} - a_{n-2} + 2 \text{ for } n \geq 3$$

Find the closed form formula for a_n.

Solution 1. By testing the first few terms, we see that $a_n = n^2$. We can prove that this is true by using induction on n.
Base Case: $n = 1, 2$ are true because $1^2 = 1$ and $2^2 = 4$
Induction hypothesis: Let $k \geq 2$. Assume that $a_i = i^2$ for all $1 \leq i \leq k$.
Induction step: We have the following:

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} + 2 \\ &= 2k^2 - (k-1)^2 + 2 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

Therefore, by induction, the result follows.

Remark. Other than guessing the formula, it is possible to actually solve this recurrence relation. For example, you can set $b_n = a_n - a_{n-1}$ to see that b_n is a linear. This means that a_n must be modeled with a quadratic. Another method is to use generating functions. Note that the usual method of solving recurrence, where you start by solving for the roots of $x^2 - 2x + 1$, does not work here because that method only work for linear recurrence without a constant term.

2. Let x and y be positive integers satisfying

$$xy = 2014x + 2014y$$

Prove that $x \leq 4058210$.

Solution. First, write the equation as

$$\frac{1}{2014} = \frac{1}{x} + \frac{1}{y}$$

Observe that we must have $y \geq 2015$ or else x will be negative. Assume that $x > 4058210 = (2014)(2015)$, then

$$\frac{1}{y} = \frac{1}{2014} - \frac{1}{x} > \frac{1}{2014} - \frac{1}{(2014)(2015)} = \frac{1}{2015}$$

Therefore, $y < 2015$, which is a contradiction.

Remark. Depending on how you arrange the initial equation, there are various ways to analyze the equation to get the result.

3. In a convex quadrilateral, the lengths of the diagonals are 100 and 1. Given that the perimeter is an integer, find all possible perimeters.

Solution. Let the vertices be A, B, C, D. Let O be the intersection of the two diagonals. By triangular inequality, we have

$$\begin{aligned} |AB| + |BC| &> 100 \\ |CD| + |DA| &> 100 \end{aligned}$$

Therefore, $|AB| + |BC| + |CD| + |DA| > 200$. By triangular inequality, we have

$$\begin{aligned} |AO| + |OB| &> |AB| \\ |BO| + |OC| &> |BC| \\ |CO| + |OD| &> |CD| \\ |DO| + |OA| &> |DA| \end{aligned}$$

Therefore, $|AB| + |BC| + |CD| + |DA| < 202$. Since the perimeter is an integer, it must be 201.

Remark. It is possible to analyze this by looking at degenerate cases. However, you must clearly explain why they provide the minimum and maximum possible perimeter.

4. Determine if there exists a polynomial f with integer coefficients such that

$$f(3) = 1, f(5) = 3, \text{ and } f(7) = 9$$

Solution. If such a polynomial exists then there must also be a polynomial such that $f(-2) = 1$, $f(0) = 3$, and $f(2) = 9$. We can write f as

$$f(x) = \sum_{i=0}^{n} a_n x^n$$

Note that $a_0 = 3$. We have the following computation

$$\begin{aligned} f(-2) + f(2) &= \sum_{n=0}^{k} a_n(-2)^n + \sum_{i=0}^{n} a_n 2^n \\ &= \sum_{n=0}^{k} a_n((-2)^n + 2^n) \\ &= \sum_{n=0, n \text{ even}}^{k} a_n(2^{2n}) \\ &= 6 + \sum_{n=2, n \text{ even}}^{k} a_n(2^{2n}) \\ &= 10 \end{aligned}$$

Therefore,

$$4 = \sum_{n=2, n \text{ even}}^{k} a_n(2^{2n})$$

If $k \geq 2$, then the right hand side is divisible by 16, but the left hand side is not. Therefore, this is not possible.
If $k = 1$, then $f(x) = a_1 x + 3$. This is not possible because the y coordinates are not evenly spaced.
If $k = 0$, then $f(x) = 3$, which is not possible.
Therefore, no such polynomial exist.

Remark. If you do not apply the initial transformation, you can analyze $f(7) - f(5) + f(3) - f(5)$ to find the contradiction.

5. Find all non-negative integer solutions to $x!^{y!} = z!$.

Solution. The solutions are

$$\begin{aligned} & \{(x, y, z) : x = z = 1, y \in \mathbb{Z}_{\geq 0}\} \\ \cup & \{(x, y, z) : x = z = 0, y \in \mathbb{Z}_{\geq 0}\} \\ \cup & \{(x, y, z) : x = 0, z = 1, y \in \mathbb{Z}_{\geq 0}\} \\ \cup & \{(x, y, z) : x = 1, z = 0, y \in \mathbb{Z}_{\geq 0}\} \\ \cup & \{(x, y, z) : x = z, y = 0\} \\ \cup & \{(x, y, z) : x = z, y = 1\} \end{aligned}$$

To see that there are no more solutions, assume that $x \geq 2$ and $y \geq 2$, then we have

$$x! < x!^2 \leq x!^{y!} = z!$$

Claim. There exists a prime p such that $x! < p! < x!^2$

Proof. Let q be the largest prime less than or equal to x. Then $x!^2$ must be divisible by q^2. Since $x!$ is only divisible by q and not q^2, then $(2q)! \leq x!^2$. By Bertrand's Postulate, there exists a prime p such that $q < p < 2q$. Thus, $q! < p! < (2q)!$. Since q was the largest prime less than or equal to x, we must have $p > x$. Therefore, $x! < p! < x!^2$.

By the claim, we know that there exists a prime p such that $x! < p! < x!^{y!} = z!$ Therefore, p must divide z. This means that p must divide $x!^{y!}$, which is not possible. Therefore, there are no more solutions.

Remark. It is possible to solve this problem with induction.