Math 4315 – PDE's

Boundary Conditions

So far we have started to solve PDE's

Ex $\quad u_x - u_y = 1$

Start with

$$u_s = u_x x_s + u_y y_s$$

& choose $\quad x_s = 1$

$\quad y_s = -1$

so $\quad u_s = 1$

We solve

$$x = s + a(r)$$
$$y = -s + b(r)$$
$$u = s + c(r)$$

$\}$ $x + y = a(r) + b(r) = A(r)$

$u + y = b(r) + c(r) = B(r)$

$$r = A^{-1}(x+y) \Rightarrow u = -y + f(x+y)$$

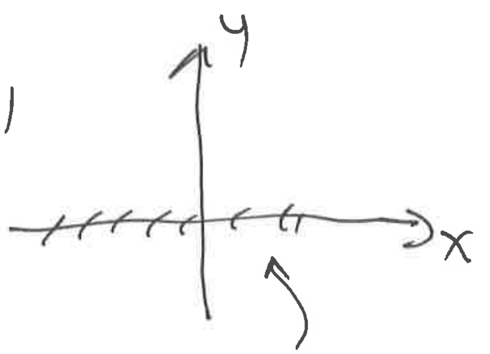
so what about the arbitrary function f?

we now attach a boundary condition, say

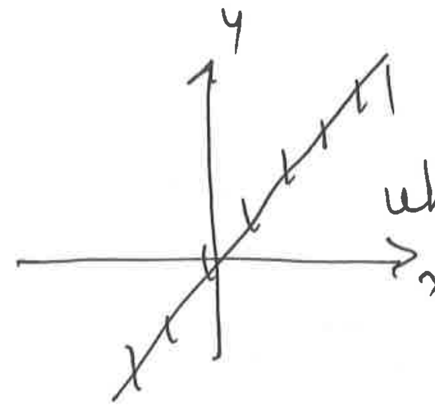
~~ex~~ (i) $u(x,0) = 2x$

(ii) $u(x,x) = 2x$

(i)



$y=0$ where $u=2x$



$y = x$
where $u = 2x$

so with these, we find the form of f.

i) with $u = -y + f(x+y)$

sub $u = 2x$, $y = 0$

$\Rightarrow \quad 2x = 0 + f(x) \Rightarrow f(x) = 2x$

$$u = -y + 2(x+y) = 2x+y$$

(ii) $y = x \quad u = 2x$

sub $\quad 2x = -x + f(2x)$

$f(2x) = 3x$

let $\quad 2x = \lambda \quad$ so $\quad x = \lambda/2$

$\Rightarrow \quad f(\lambda) = \frac{3\lambda}{2} \quad \leftarrow$ this det f

sol^n $\quad u = -y + \frac{3}{2}(x+y)$

$= \frac{3}{2}x + y/2$

& you can check each satisfies the PDE

& $\quad u_1(x,0) = 2x + 0 = 2x \quad \checkmark$

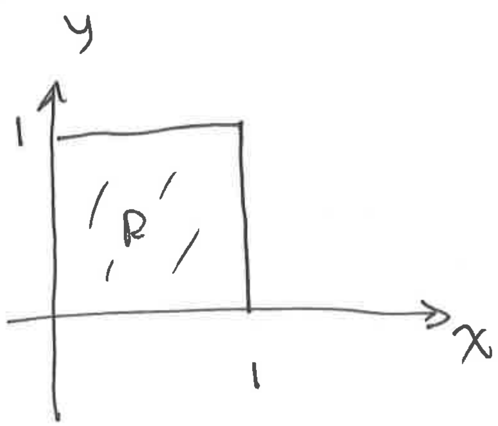
$u_2(x,x) = \frac{3x}{2} + \frac{x}{2} = 2x \quad \checkmark$

So to find the sol^n subject to the B.C. we need to find the general sol^n before using the B.C. Our examples so far have been relatively easy (they can become hard)

So we try to pass the B.C. from (x,y) to the (r,s) plane. Before doing this consider the change of variables

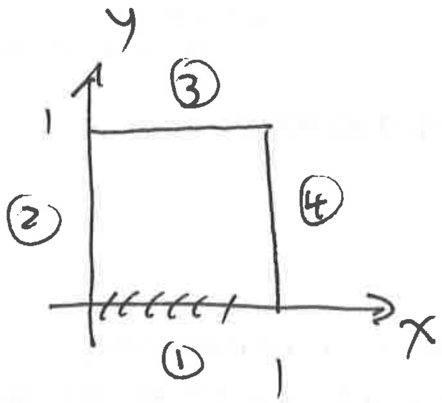
$r = x+y$ let us determine what
$s = y$ happens to the xy plane

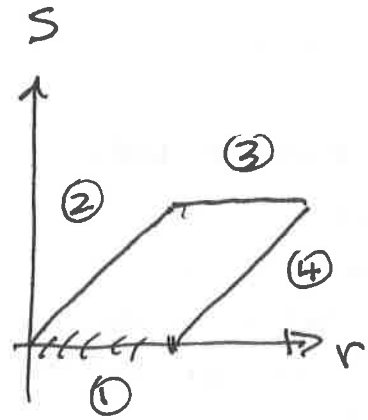
and in particular the following box



the box is bound by 4 lines

$x = 0, 1, \quad y = 0, 1$





(1) $y=0 \quad 0 \le x \le 1$ so $r=x, s=0$

(2) $x=0 \quad 0 \le y \le 1$ so $r=y, s=y \Rightarrow s=r$

(3) $y=1 \quad 0 \le x \le 1$ so $r=x+1, s=1$

(4) $x=1 \quad 0 \le y \le 1$ so $r=y+1, s=y \Rightarrow s=r-1$

and you can show $(x,y) = (\frac{1}{2}, \frac{1}{2}) \Rightarrow (r,s) = (1, \frac{1}{2})$

a pt inside goes inside

the main thing is

(x,y) plane boundary $y=0$
(r,s) plane boundary $s=0$

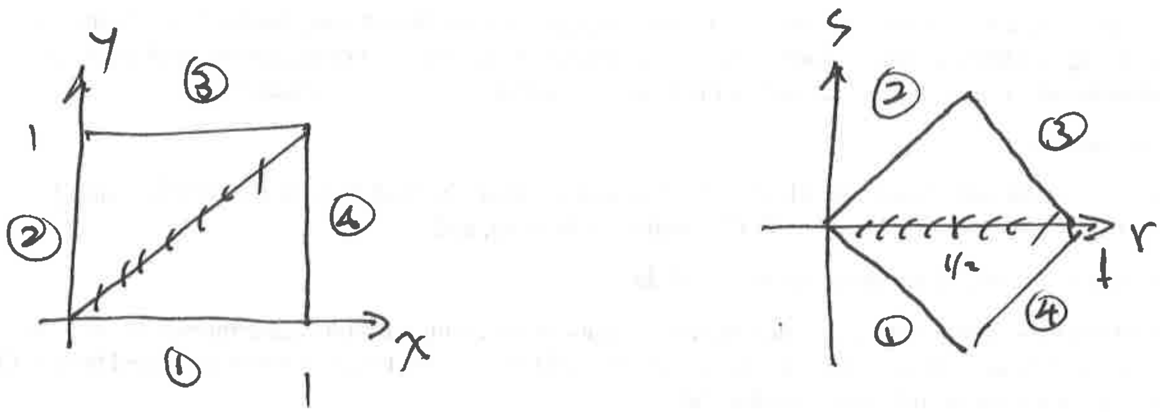
boundary → boundary $x=r$

this connects the boundaries

(ii) consider the change of variables

$$r = \frac{x+y}{2} \qquad s = \frac{y-x}{2}$$

and the same square in addition to $y = x$



(1) $0 \le x \le 1 \quad y = 0$ $\quad r = x/2 \quad s = -x/2 \Rightarrow s = -r$

(2) $x = 0 \quad 0 \le y \le 1$ $\quad r = y/2 \quad s = y/2 \Rightarrow s = r$

(3) $0 \le x \le 1 \quad y = 1$ $\quad r = \frac{x+1}{2} \quad s = \frac{1-x}{2} \Rightarrow r+s = 1$

(4) $x = 1 \quad 0 \le y \le 1$ $\quad r = \frac{1+y}{2} \quad s = \frac{y-1}{2} \Rightarrow r-s = +1$

and the boundary $y = x$

$r = x \quad \&, \quad s = 0$

like we saw in the previous ex.

so when solving

$$u_x - u_y = 1 \qquad u(x,0) = 2x$$
$$u(x,x) = 2x$$

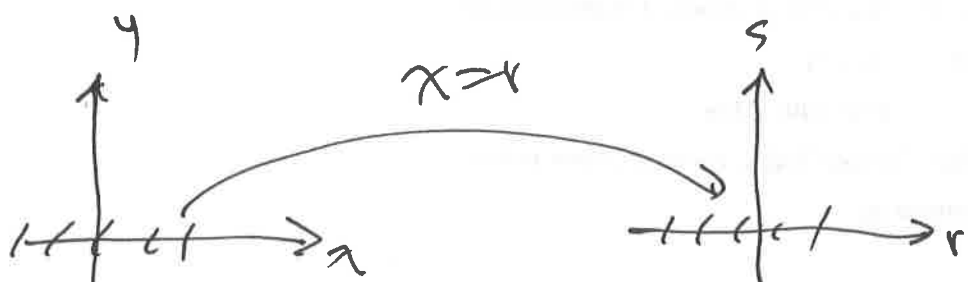
we again use $u_s = u_x x_s + u_y y_s$

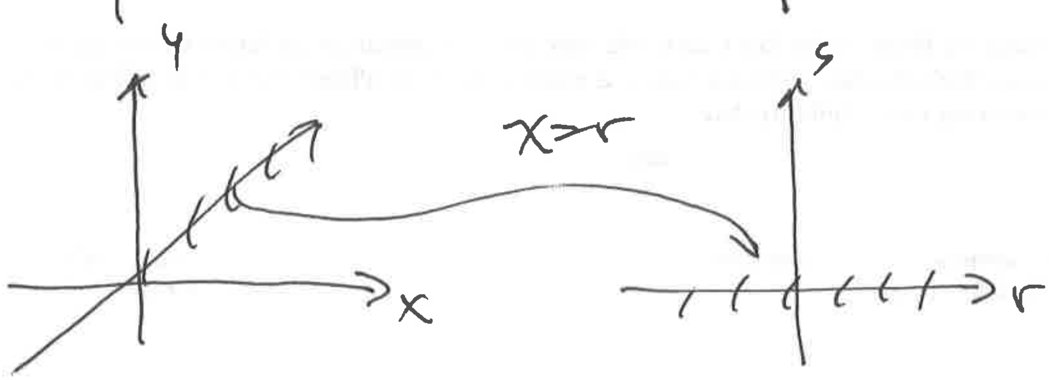
pick $x_s = 1$

$y_s = -1$ we are now in

$u_s = 1$ (r,s) coord's

but now create a new boundary in (r,s) and have this $s=0$ and connect the 2 boundaries by $x=r$





(i) $u(x,0) = 2x$

so on $s=0$ $x=r,\ y=0,\ u=2x=2r$

Solve $x_s = 1$ subject to $s=0$

$y_s = -1$ $x=r$

$u_s = 1$ $y=0$

$u=2r$

(i) $x = s + a(r)$ $s=0, x=r \Rightarrow a(r)=r$

so $\boxed{x = s+r}$

ii) $y = -s + b(r)$ $s=0,\ y=0 \Rightarrow b(r)=0$

so $\boxed{y=-s}$

(iii) $u = s + c(r)$ $s=0\ u=2r \Rightarrow c(r)=2r$

so $\boxed{u = s+2r}$

$s=-y,\ r = x-s = x+y$ $u = -y + 2(x+y)$

$\boxed{u = 2x+y \text{ sol}^n}$

ii) $u(x,x) = 2x$

on $s=0$ $x=r,\ y=r,\ u=2r$

Solve $x_s = 1,\ y_s = -1,\ u_s = 1$

$x = s + a(r)$ $s=0$ $x=r \Rightarrow a(r)=r$

$$\boxed{x = s+r}$$

$y = -s + b(r)$ $s=0$ $y=r \Rightarrow b(r)=r$

$$\boxed{y = -s+r}$$

$u = s + c(r)$ $s=0$ $u=2r \Rightarrow c(r)=2r$

$$\boxed{u = s+2r}$$

so $x+y = 2r,\ x-y = 2s \Rightarrow r = \frac{x+y}{2}\ s = \frac{x-y}{2}$

$u = s+2r = \frac{x-y}{2} + x+y = \frac{3}{2}x + y/2$ // sol^n