Math 4381 / 6378

Consider
$$y'' = \frac{y'^4}{xy^2(xy'-y)}$$

this ODE has 2 symmetries with infinitesimals

$$X = c_1 xy \quad Y = c_1 y^2 + c_2 y$$

we saw last class that solving

$$X r_x + Y r_y = 0 \quad X s_x + Y s_y = 1$$

would lead to new variables r & s and that the new ODE is independent of s

The question we ask - which symmetry should I use and what happens to the other symmetry? As there are 2 sym. in this example, we will use one and see what happens to the other.

Sym & we use

$$X_1 = xy, \quad Y_1 = y^2$$

and see what happens to

$$X_2 = 0, \quad Y_2 = y$$

Cofr $\quad xy r_x + y^2 r_y = 0 \qquad xy s_x + y^2 s_y = 1$

$$r = A\left(\frac{y}{x}\right) \qquad s = -\frac{1}{y} + B(y/x)$$

If $r = y/x \quad s = -\frac{1}{y}$

& we get the new ODE

$$r s'' + 2s' + r^4 s'^4 = 0$$

If we let $u = s', \; u' = s''$

and this 2nd order ODE becomes

$$r u' + u + r^4 u^4 = 0 \qquad \text{Bernoulli}$$

So we can associate an infinitesimal operator to go with the infinitesimal trans

$\bar{x} = x + \varepsilon X, \quad \bar{y} = y + \varepsilon Y, \quad \bar{r} = r + \varepsilon R, \quad \bar{s} = s + \varepsilon \$$

as $\Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + R \frac{\partial}{\partial r} + \$ \frac{\partial}{\partial s}$

so if $r = \frac{y}{x} \quad s = -\frac{1}{y}$

then $R = \frac{Y}{x} - \frac{y}{x^2} X, \quad \$ = \frac{Y}{y^2}$

& if $X_2 = 0, \ Y_2 = y$

then $R_2 = \frac{y}{x} = r \quad \$_2 = \frac{1}{y} = -s$

Since we defined $u = s'$ then

$$\theta = \$[r] = \$_r + (\$_s - R_r) s' - R_s s'^2 \quad \left(\begin{matrix}\text{extended}\\\text{trans}\end{matrix}\right)$$

$$= 0 + (-1 - 1) s' + 0$$

$$= -2u$$

so $R_2 = r \quad \theta = -2u \leftarrow$ it passes to new 1st order ODE.

Sym 2 $X_2 = 0, \; Y_2 = y$

Cof r $y r_y = 0 \qquad y s_y = 1$

$\Rightarrow r = A(x) \qquad s = \ln y + B(x)$

choose $r = x, \; s = \ln y$

so $x = r, \; y = e^{s}$

and new ODE is

$$s'' = \frac{(s'^2 - r^2 s' + r)\, s'^2}{r(rs' - 1)}$$

and if $s' = u, \; s'' = u'$ and this becomes

$$u' = \frac{(u^2 - r^2 u + r)\, u^2}{r(ru - 1)}$$

Now let's see what ~~happens~~ to the other symmetry.

if $\Gamma = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + R\frac{\partial}{\partial r} + S\frac{\partial}{\partial s}$

if $r = x, \quad s = \ln y$

$R = X, \quad S = \frac{Y}{y}$

the other sym $X = xy \quad Y = y^2$

so $R = xy \quad S = \frac{y^2}{y} = y$

so $x = r, \quad y = e^s$

so these becomes

$R = re^s, \quad S = e^s$

Now $u = s' \Rightarrow \sigma = S_r + (S_s - R_r)s' - R_s s'^2$

$= 0 + (e^s - e^s)s' - re^s s'^2$

so $\sigma = -re^s u^2$

Now $u = s'$ so $s = \int u\,dr \quad \sigma = -re^{\int u\,dr} u^2$ ← a nonlocal sym.

Question – Is it possible to determine ahead of time which sym. to use and which one will pass

If Γ_i & Γ_j are 2 sym. generator

if we define the Lie Bracket as

$$[\Gamma_i, \Gamma_j] = \Gamma_i(\Gamma_j) - \Gamma_j(\Gamma_i)$$

then if

$$[\Gamma_i, \Gamma_j] = k\Gamma_i, \quad k \text{ constant}$$

we should use Γ_i for the reduction and the symmetry associated with Γ_j will pass

so for our example

$$\Gamma_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \quad \& \quad \Gamma_2 = y\frac{\partial}{\partial y}$$

$$\Gamma_1(\Gamma_2(u)) = \left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)\left(y\frac{\partial u}{\partial y}\right)$$

$$= xy^2 u_{xy} + y^3 u_{yy} + y^2 u_y$$

$$\Gamma_2(\Gamma_1(u)) = y\frac{\partial}{\partial y}\left[xy u_x + y^2 u_y\right]$$

$$= xy^2 u_{xy} + xy\, u_x + y^3 u_{yy} + 2y^2 u_y$$

$$\Gamma_1(\Gamma_2 u) - \Gamma_2(\Gamma_1(u)).$$

$$= \cancel{xy^2 u_{xy}} + \cancel{y^3 u_{yy}} + y^2 u_y - \cancel{xy^2 u_{xy}} - xy u_x$$

$$- \cancel{y^3 u_{yy}} - 2y^2 u_y$$

$$= -xy\, u_x - y^2 u_y$$

$$= -\Gamma_1(u)$$

so use Γ_1 & Γ_2 will poss.