Math 4315 - PDE's

Lecture 8
Sept. 16, 16

Ex

$$y u_x - x u_y = 0$$

$x_s = y$
$y_s = -x$
$u_s = 0$

From the notes from the previous class, the process of solving for x & y is difficult and getting rid of r & s tricky.

So we'll introduce a new way to do this

In this example

$$\frac{\partial x}{\partial s} = y, \quad \frac{\partial y}{\partial s} = -x$$

or $\quad \dfrac{\partial x}{y} = \partial s = \dfrac{\partial y}{-x} \quad$ or $\quad \dfrac{\partial x}{y} = -\dfrac{\partial y}{x} \quad$ no s

so $\quad x\,\partial x + y\,\partial y = 0$

$\Rightarrow \quad \dfrac{x^2}{2} + \dfrac{y^2}{2} = \dfrac{A(r)}{2} \quad$ note: r is still here

$$\frac{\partial u}{\partial s} = 0 \Rightarrow \quad u = C(r)$$

Now we are really treating A(r) as const
c(r)

$x^2+y^2 = c_1, \quad u = c_2$

so $u = f(x^2+y^2)$ or $c_2 = f(c_1)$

so we can do this in general

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

$x_s = a, \quad y_s = b \quad u_s = c$

$$\frac{\partial x}{a} = \partial s \qquad \frac{\partial y}{b} = \partial s \qquad \frac{\partial u}{c} = \partial s$$

$$\Rightarrow \frac{\partial x}{a(x,y,u)} = \frac{\partial y}{b(x,y,u)} = \frac{\partial u}{c(x,y,u)}$$

and since when we integrate we treat r as const

then

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}$$ ← characteristic Equations.

let's look at some previous example

From lecture 4

$$u_x + u_y = 2 \qquad u(x,0) = x^2$$

CE. $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{du}{2}$

Pick pairs (1)st $dx = dy \Rightarrow x = y + c_1$

(2)nd $2dy = du \Rightarrow 2y = u - C_2$

so $c_1 = x - y \qquad C_2 = u - 2y$

sol^n $C_2 = f(c_1) \Rightarrow \qquad u - 2y = f(x-y)$

or $\qquad u = 2y + f(x-y)$

Now B.C. $u(x,0) = x^2 \Rightarrow \qquad x^2 = 0 + f(x)$

$$f(x) = x^2$$

$$u = 2y + (x-y)^2$$

Ex Lecture 2 $\quad x u_x - y u_y = 2u$

CE. $\quad \dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{du}{2u}$

1st $\quad \dfrac{dx}{x} = \dfrac{dy}{-y} \Rightarrow \ln x = -\ln y + \ln c_1$

$$xy = c_1$$

2nd $\quad \dfrac{2\,dx}{x} = \dfrac{du}{u} \Rightarrow 2\ln|x| = \ln|u| - \ln C_2$

$$C_2 = u/x^2$$

Sol^n $\quad C_2 = f(C_1) \Rightarrow \dfrac{u}{x^2} = f(xy)$

or $\quad u = x^2 f(xy)$ which we saw

Ex $\quad x u_x + (x+y) u_y = 1$ $\quad$ lecture 5

CE $\quad \dfrac{dx}{x} = \dfrac{dy}{x+y} = \dfrac{du}{1}$

1st $\quad \dfrac{dx}{x} = \dfrac{dy}{x+y}$ or $\dfrac{dy}{dx} = \dfrac{x+y}{x} = 1 + y/x$ homo

$v = y/x$ so $y = xv$ $\qquad x v' + \cancel{v} = 1 + \cancel{v}$

$y' = xv' + v$ $\quad \Rightarrow \quad v' = \dfrac{1}{x} \Rightarrow v = \ln|x| + c_1$

$$y/x - \ln|x| = c_1$$

2nd pair $\quad \frac{dx}{x} = du$

$$\ln|x| = u - c_2$$

$$\Rightarrow \quad c_2 = u - \ln|x|$$

sol^n $\quad c_2 = f(c_1) \Rightarrow \quad u - \ln|x| = f\left(y/x - \ln|x| \right)$

or $\quad u = \ln|x| + f\left(y/x - \ln|x| \right)$

in that ex there was a B.S. $\quad u(x,x) = 2x+1$

so sub into sol^n

$$2x+1 = \ln|x| + f(1 - \ln|x|)$$

if

$$\lambda = 1 - \ln|x| \Rightarrow \ln|x| = 1 - \lambda \Rightarrow x = e^{1-\lambda}$$

so $\quad 2 \cdot e^{1-\lambda} + 1 - (1-\lambda) = f(\lambda)$

$$f(\lambda) = 2e^{1-\lambda} + \lambda$$

sol^n $\quad u = \cancel{\ln|x|} + 2e^{1 - y/x + \ln|x|} + y/x - \cancel{\ln|x|}$

$$= y/x + 2e^{1 - y/x + \ln|x|}$$