

**Chapter 4**  
**Polynomial Functions**

**Section 4-6**  
**The Fundamental Theorem of Algebra**

## The Fundamental Theorem of Algebra

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution  $x = -1$  is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial  $f(x)$  and the number of solutions of  $f(x) = 0$ . This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777–1855).

### Core Concept

#### The Fundamental Theorem of Algebra

**Theorem** If  $f(x)$  is a polynomial of degree  $n$  where  $n > 0$ , then the equation  $f(x) = 0$  has at least one solution in the set of complex numbers.

**Corollary** If  $f(x)$  is a polynomial of degree  $n$  where  $n > 0$ , then the equation  $f(x) = 0$  has exactly  $n$  solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

The corollary to the Fundamental Theorem of Algebra also means that an  $n$ th-degree polynomial function  $f$  has exactly  $n$  zeros.

### EXAMPLE 1 Finding the Number of Solutions or Zeros

- How many solutions does the equation  $x^3 + 3x^2 + 16x + 48 = 0$  have?
- How many zeros does the function  $f(x) = x^4 + 6x^3 + 12x^2 + 8x$  have?

Remember that the number of solutions, number of zeros, and the number of  $x$ -intercepts all mean the same thing.

- ▶ How many solutions does the equation  $x^4 + 7x^2 - 144 = 0$  have?
- ▶ How many zeros does the function  $f(x) = x^3 - 5x^2 - 8x + 48$  have?

## Complex Conjugates

Pairs of complex numbers of the forms  $a + bi$  and  $a - bi$ , where  $b \neq 0$ , are called **complex conjugates**. In Example 2, notice that the zeros  $2i$  and  $-2i$  are complex conjugates. This illustrates the next theorem.

## Core Concept

### The Complex Conjugates Theorem

If  $f$  is a polynomial function with real coefficients, and  $a + bi$  is an imaginary zero of  $f$ , then  $a - bi$  is also a zero of  $f$ .



## EXAMPLE 2 Finding the Zeros of a Polynomial Function

Find all zeros of  $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$ .

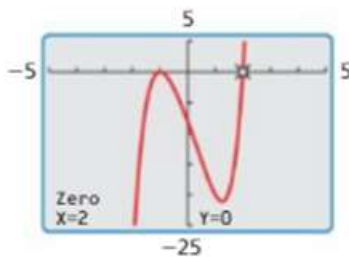
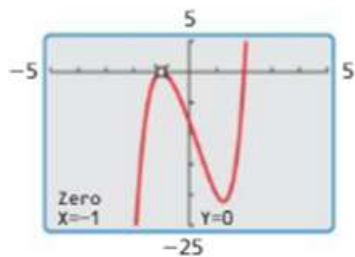
### SOLUTION

#### STUDY TIP

Notice that you can use imaginary numbers to write  $(x^2 + 4)$  as  $(x + 2i)(x - 2i)$ . In general,  $(a^2 + b^2) = (a + bi)(a - bi)$ .



The graph of  $f$  and the real zeros are shown. Notice that only the *real* zeros appear as  $x$ -intercepts. Also, the graph of  $f$  touches the  $x$ -axis at the repeated zero  $x = -1$  and crosses the  $x$ -axis at  $x = 2$ .



Find all zeros of the polynomial function.

▶ 3.  $f(x) = x^3 + 7x^2 + 16x + 12$

▶ 4.  $f(x) = x^5 - 3x^4 + 5x^3 - x^2 - 6x + 4$

**EXAMPLE 3** Using Zeros to Write a Polynomial Function

Write a polynomial function  $f$  of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and  $3 + i$ .

# Descartes's Rule of Signs

French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

## Core Concept

### Descartes's Rule of Signs


Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$  be a polynomial function with real coefficients.

- The number of *positive real zeros* of  $f$  is equal to the number of changes in sign of the coefficients of  $f(x)$  or is less than this by an even number.
- The number of *negative real zeros* of  $f$  is equal to the number of changes in sign of the coefficients of  $f(-x)$  or is less than this by an even number.

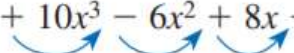
### EXAMPLE 4 Using Descartes's Rule of Signs

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for  $f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$ .

#### SOLUTION

$$f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8.$$


The coefficients in  $f(x)$  have **3 sign changes**, so  $f$  has 3 or 1 positive real zero(s).

$$\begin{aligned} f(-x) &= (-x)^6 - 2(-x)^5 + 3(-x)^4 - 10(-x)^3 - 6(-x)^2 - 8(-x) - 8 \\ &= x^6 + 2x^5 + 3x^4 + 10x^3 - 6x^2 + 8x - 8 \end{aligned}$$


The coefficients in  $f(-x)$  have **3 sign changes**, so  $f$  has 3 or 1 negative zero(s).

► The possible numbers of zeros for  $f$  are summarized in the table below.

Positive real zeros	Negative real zeros	Imaginary zeros	Total zeros
3	3	0	6
3	1	2	6
1	3	2	6
1	1	4	6

### EXAMPLE 5 Real-Life Application



A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed  $x$  (in hundreds of RPMs) of the engine shaft and the speed  $s$  (in miles per hour) of the boat are modeled by

$$s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11.0.$$

What is the tachometer reading when the boat travels 15 miles per hour?

**Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.**

9.  $f(x) = x^3 + 9x - 25$

10.  $f(x) = 3x^4 - 7x^3 + x^2 - 13x + 8$

11. **WHAT IF?** In Example 5, what is the tachometer reading when the boat travels 20 miles per hour?