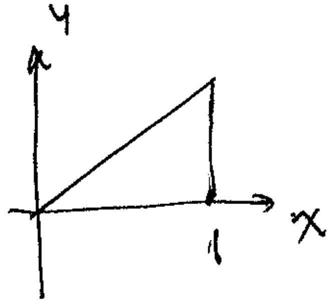
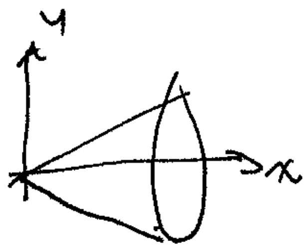
Math 1496 – Calc I

Consider the following. The line $y = x$ on $[0,1]$



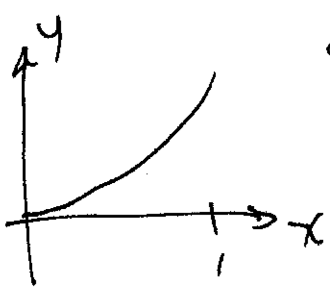
Let us revolve this line about the x axis so we obtain a cone. We know the volume



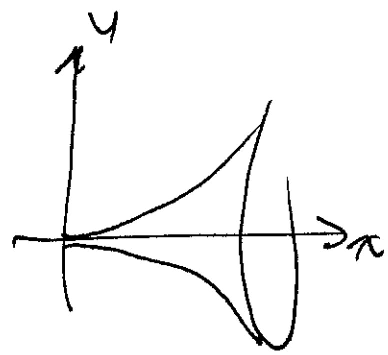
is $V = \frac{1}{3}\pi r^2 h$

& here $r = 1$, $h = 1$ so $V = \frac{1}{3}\pi$

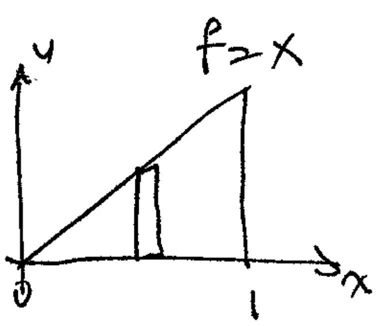
Now consider $y = x^2$



& do the same



Now we obtain a horn but what is the volume of this? As we did with area & approximating w/ rectangles we do the same but with disc's.



we subdivide interval into n pieces so $\Delta x = \frac{1}{n}$. we choose the right end pt of i^{th} rectangle

$$f(c_i) = \frac{i}{n}$$

Now we rotate the rectangle about the x axis

the volume is $\pi \underset{\Delta x}{r^2} = \pi \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$

$$\Delta V = \frac{\pi i^2}{n^3}$$

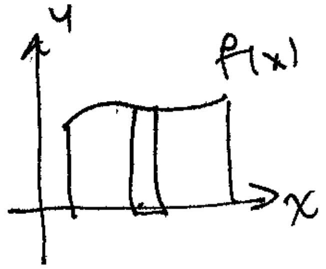
Now add up discs & $n \to \infty$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi\left(\frac{i}{n}\right)^2 \frac{1}{n} = \int_0^1 \pi x^2 dx$$

$$= \left.\frac{\pi x^3}{3}\right|_0^1 = \pi/3 \quad \text{same answer as before}$$

so now we will do this in general. However we will derive the formula bypassing the Riemann Sum (well it's really there)

Volumes of Revolution



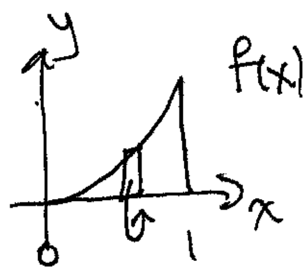
$f(x)$ thickness dx

height (radius) $r_i = \pi f(x)^2$

$dV = \pi f^2(x)\,dx$

then add them up $V = \int_a^b \pi f^2(x)\,dx$

Ex2

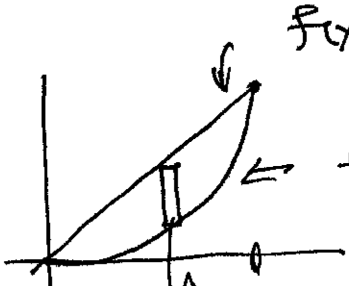


$f(x) = x^2$

$$V = \pi \int_0^1 (x^2)^2\,dx = \int_0^1 \pi x^4\,dx$$

$$= \frac{\pi x^5}{5}\Big|_0^1 = \pi/5$$

Two curves



$f(x) = x$

$f(x) = x^2$

intersection pt

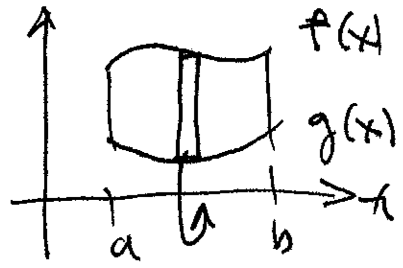
$x^2 = x$

$x(x-1) = 0 \quad x = 0, 1$

$$V = \pi \int_0^1 x^2\,dx - \pi \int_0^1 (x^2)^2\,dx = \pi \frac{x^3}{3}\Big|_0^1 - \pi \frac{x^5}{5}\Big|_0^1$$

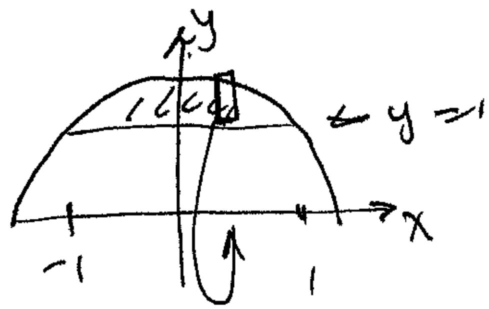
$$= \frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}$$

In general



$$V = \pi \int_a^b f^2(x) - g^2(x)\, dx$$

Ex4 Find volume of revolution of the region bound between the $y = \sqrt{2-x^2}$ & $y = 1$



we need intersection pt

so $\sqrt{2-x^2} = 1$

$\Rightarrow 2 - x^2 = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

So $\pi \int_{-1}^{1} \left(\sqrt{2-x^2}\right)^2 - 1^2\, dx$

$= \pi \int_{-1}^{1} (2 - x^2 - 1)\, dx$

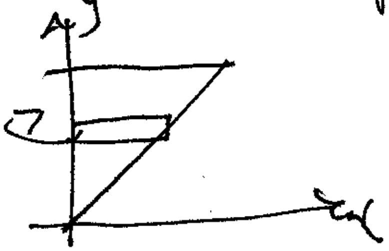
$= \pi \int_{-1}^{1} (1 - x^2)\, dx$

$= \pi \left(x - \frac{x^3}{3} \Big|_{-1}^{1} \right) = \pi \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right]$

$= \pi \cdot \frac{4}{3}$

Revolutions about y axis

Now we consider $y=x$ and $y=1$ & revolve this region about y axis. Here we will use horizontal rectangles so the radius of the disc



is $x = f(y) = y$

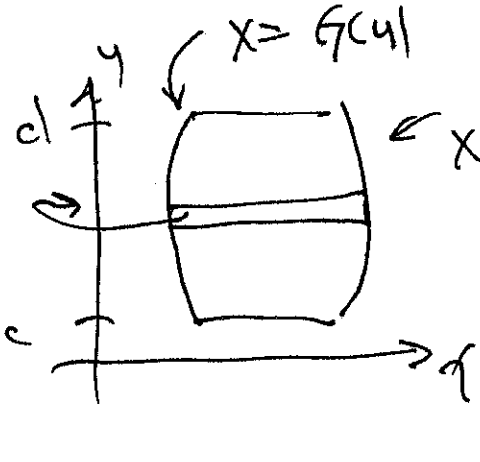
$A_i = \pi y^2 dy$

radius is now in the x → direction

$$V = \pi \int_0^1 y^2 dy = \pi \frac{y^3}{3}\Big|_0^1 = \pi/3$$

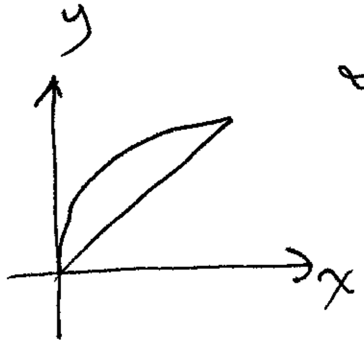
we see this is again just a cone $V = \frac{1}{3}\pi r^2 h = \pi/3$

In general with 2 curves



$$V = \pi \int_c^d F^2(y) - G^2(y)\, dy$$

Ex Revolution the region $y=\sqrt{x}$ & $y=x$ about the x & y axis

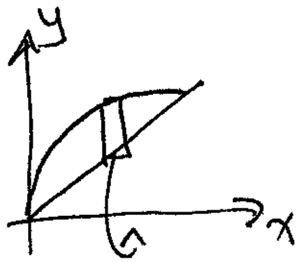


Intersection points

$\sqrt{x}=x$ so $x=x^2 \Rightarrow x=0,1$

so $y=0,1$

About x axis



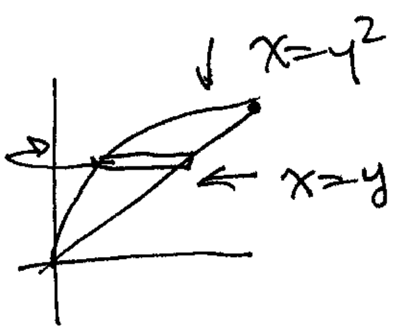
$$V=\pi\int_0^1 (\sqrt{x})^2-x^2\,dx$$

$$=\pi\int_0^1 (x-x^2)\,dx=\pi\left(\frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1\right)$$

$$=\pi\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\pi/6$$

About y axis



$$\pi\int_0^1 (y^2-y^4)\,dy=\pi\left(\frac{y^3}{3}-\frac{y^5}{5}\right)\Big|_0^1$$

$$=\pi\cdot\frac{2}{15}$$