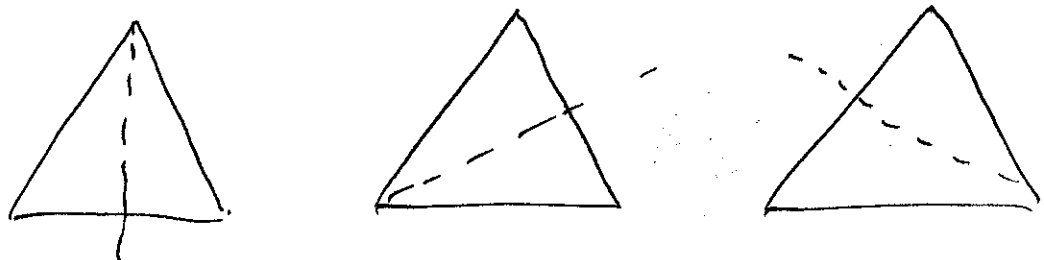
Symmetry is all through nature and in mathematics can be a useful tool in exploring mathematics. Examples are reflections in

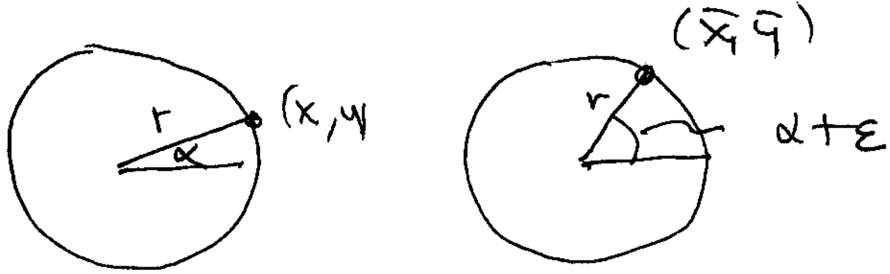
3 axes



in addition to rotations of 120°. These are descrete symmetries. The circle is highly symmetric in rotation and is a continuous symmetr.

So invariance is we make a change and the object remains the same.

so how do we do this mathematically



so $x = r\cos\alpha$, $y = r\sin\alpha$

$\bar{x} = r\cos(\alpha+\varepsilon)$, $\bar{y} = r\sin(\alpha+\varepsilon)$

$= r\sin\alpha\cos\varepsilon + r\cos\alpha\sin\varepsilon$

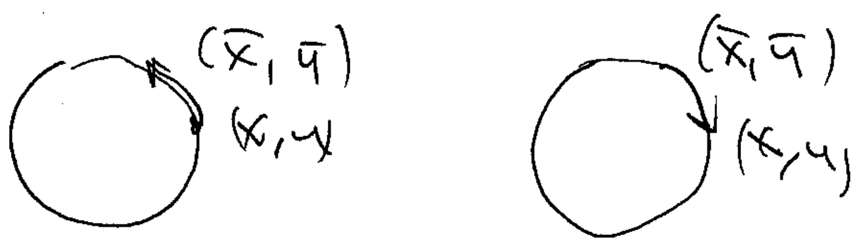
$= r\cos\alpha\cos\varepsilon$

$- r\sin\alpha\sin\varepsilon$

$\bar{x} = x\cos\varepsilon - y\sin\varepsilon$ $\quad \bar{y} = y\cos\varepsilon + x\sin\varepsilon$

<u>Note:</u> $\varepsilon \to 0$ $\quad \bar{x} = x,\ \bar{y} = y$

inverse



rotate α then β a $(\alpha+\beta)$

These properties form a group.

show under rotation the circle is invariant

show $\bar{x}^2+\bar{y}^2=r^2$ if $x^2+y^2=r$

so $\bar{x}^2+\bar{y}^2 = (x\cos\varepsilon - y\sin\varepsilon)^2$
$+(y\cos\varepsilon + x\sin\varepsilon)^2$

$= x^2\cos^2\varepsilon - \cancel{2xy\sin\varepsilon\cos\varepsilon} + y^2\sin^2\varepsilon$
$y^2\cos^2\varepsilon + \cancel{2xy\sin\varepsilon\cos\varepsilon} + x^2\sin^2\varepsilon$

$= (x^2+y^2)(\sin^2\varepsilon+\cos^2\varepsilon)$

$= x^2+y^2$

so if $x^2+y^2=r^2 \Rightarrow \bar{x}^2+\bar{y}^2=r^2$ ✓

Scaling symmetry

show $y=mx$ is invariant under

$\bar{x}=e^{\varepsilon}x,\ \bar{y}=e^{\varepsilon}y$

$$\bar{y} = \bar{m}\bar{x} \Rightarrow \cancel{e^{\varepsilon}} y = m \cancel{e^{\varepsilon}} x \Rightarrow y = mx \checkmark$$

Translational Symmetry

Show $y = mx + b$ is invariant under

$$\bar{x} = x + p\varepsilon, \quad \bar{y} = y + q\varepsilon \qquad p, q \text{ const}$$

$$\bar{y} = m\bar{x} + b \text{ so } y + q\varepsilon = m(x + p\varepsilon) + b$$

$$y = mx + b + (mp - q)\varepsilon$$

So if we want invariance we need

$$mp = q$$

so $\bar{x} = x + p\varepsilon, \quad \bar{y} = y + mp\varepsilon$

Does this apply to differential Eq^n's ?

Consider

$$\frac{dy}{dx} = xy \quad \& \quad \bar{x} = e^{\varepsilon}x, \; \bar{y} = e^{\varepsilon}y \quad L$$

Called "Lie Group"

Is this inv. under L

ie $\frac{d\bar{y}}{d\bar{x}} = \bar{x}\bar{y}$ if $\frac{dy}{dx} = xy$?

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\cancel{e^{\varepsilon}}\frac{dy}{dx}}{\cancel{e^{\varepsilon}}\frac{dx}{dx}} = \frac{dy}{dx}$$

$$\bar{x}\bar{y} = e^{\varepsilon}x \cdot e^{\varepsilon}y = e^{2\varepsilon}xy$$

so is $\frac{d\bar{y}}{d\bar{x}} = \bar{x}\bar{y}$ if $\frac{dy}{dx} = xy$

$$\frac{dy}{dx} = e^{2\varepsilon}xy \quad \text{NO}$$

what about

$$\bar{x} = e^{a\varepsilon} x, \quad \bar{y} = e^{b\varepsilon} y$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{e^{b\varepsilon}\frac{dy}{dx}}{e^{a\varepsilon}\frac{dx}{dx}} = e^{(b-a)\varepsilon} y'$$

$$\bar{x}\bar{y} = e^{(a+b)\varepsilon} x y$$

$$e^{(b-a)\varepsilon} y' = e^{(a+b)\varepsilon} x y \quad \text{if } y' = xy$$

we need $b - a = a + b \Rightarrow a = 0$

so the ode is invariant under

$$\bar{x} = x, \quad \bar{y} = e^{b\varepsilon} y \qquad b \text{ arb}$$

show $\dfrac{dy}{dx} = \dfrac{(xy-1)^2}{y}$

Note

under $\bar{x} = x + \varepsilon, \quad \bar{y} = \dfrac{y}{\varepsilon y + 1}$

$\bar{x}\big|_{\varepsilon=0} = x$

$\bar{y}\big|_{\varepsilon=0} = y$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} = \frac{\frac{y'(\varepsilon y+1) - \varepsilon y' \cdot y}{(\varepsilon y+1)^2}}{1} = \frac{y'}{(\varepsilon y+1)^2}$$

RS. $$\frac{(\bar{x}\bar{y}-1)^2}{\bar{y}} = \frac{\left(\frac{(x+\varepsilon)\,y}{\varepsilon y+1} - 1\right)^2}{\frac{y}{\varepsilon y+1}} = \frac{\left(\frac{xy + \cancel{\varepsilon y} - \cancel{\varepsilon y} - 1}{\varepsilon y+1}\right)^2}{\frac{y}{\varepsilon y+1}}$$

$$= \frac{(xy-1)^2}{y(\varepsilon y+1)}$$

So $\dfrac{d\bar{u}}{d\bar{x}} = \dfrac{(\bar{x}\bar{y}-1)^2}{\bar{y}} \Rightarrow \dfrac{y'}{\cancel{\varepsilon y+1}} = \dfrac{(xy-1)^2}{y(\cancel{\varepsilon y+1})}$

$\Rightarrow \quad y' = \dfrac{(xy-1)^2}{y}$ ✓ So how is this useful?