Sept 23/16

Math 4315 PDEs

Given $a u_x + b u_y = c$ — (1)

if $u_s = u_x x_s + u_y y_s$

then picking $x_s = a$, $y_s = b$ gives $u_s = c$ — (2)

A system of 3 PDEs (well really ODEs as r is missing)

so for example

$$u_x + 2u_y = u$$

then $x_s = 1$, $y_s = 2$ so $u_s = u$

these we can integrate and then eliminate r & s

So what if the PDE is

$u_x u_y = u$

or $u_x^2 + u_y^2 = 1$

$\Big\}$ these are fully nonlinear and what do we do?

the general 1^{st} order PDEs are ~~the~~ of the form

$$F(x, y, u, u_x, u_y) = 0 \qquad -(3)$$

we will let $p = u_x$ & $q = u_y$

From our 1^{st} PDE eqn (1)

$$F = ap + bq - c$$

If we treat x, y, u, p, q as all independent variables

then $F_p = a$, $F_q = b$

and
$$\begin{aligned} c &= ap + bq \\ &= pF_p + qF_q \end{aligned}$$

so (2) becomes

$$\left.\begin{aligned} x_s &= F_p \\ y_s &= F_q \\ u_s &= pF_p + qF_q \end{aligned}\right\}$$

these work for any PDEs of the form (3)

so if our PDE is

$$u_x u_y = 1 \quad — (4)$$

then $F = pq - 1$

$F_p = q \quad F_q = p$

&

$x_s = F_p = q$

$y_s = F_q = p$

from (4)

$u_s = pF_p + qF_q = p \cdot q + q \cdot p = 2pq = 2$

and so we have 3 PDEs (ODEs) to solve.

Problem is we need p & q to do this so we'll need 2 more eq^n's

$x_s = q$

$y_s = p$

$u_s = 2$

$p_s = ?$

$q_s = ?$

Chain Rule

$$p_s = p_x x_s + p_y y_s$$

$$= p_x F_p + p_y F_q$$

$$= p_x F_p + q_x F_q$$

$\because p_y = u_{xy} = u_{yx} = q_x$

Similarly

$$q_s = q_x x_s + q_y y_s$$

$$= q_x F_p + q_y F_q \qquad \because \; q_x = u_{yx} = u_{xy} = p_y$$

$$= p_y F_p + q_y F_q$$

Now From (1) $\quad F(x, y, u, p, q) = 0$

Differentiating wrt x gives

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0$$

$$\Rightarrow \quad F_p p_x + F_q q_x = -F_x - pF_u \qquad (u_x = p)$$

and wrt y

$$F_y + F_u u_y + F_p p_y + F_q q_y = 0$$

$$\Rightarrow \quad F_p p_y + F_q q_y = -F_y - qF_u$$

So

$$\boxed{\begin{aligned} p_s &= -F_x - pF_u \\ q_s &= -F_y - qF_u \end{aligned}}$$

the last 2 eq^{n}s we need.

so for a given PDE

$$F(x, y, u, u_x, u_y) = 0$$

we need to solve (often referred to as characteristic eq^n's)

$$x_s = F_p$$
$$y_s = F_q$$
$$u_s = pF_p + qF_q$$
$$p_s = -F_x - pF_u$$
$$q_s = -F_y - qF_u$$

Ex 1 $\quad u_x u_y = 1 \quad u(x,0) = x$

so $\quad F = pq - 1$

$$F_x = F_y = F_u = 0 \quad F_p = q \quad F_q = p$$

so the CEs are

$$x_s = q$$
$$y_s = p$$
$$u_s = 2pq = 2 \quad (\text{from PDE})$$
$$p_s = 0$$
$$q_s = 0$$

Now we solve

$$p_s = a(r) \qquad q = b(r)$$

$$x_s = b(r) \Rightarrow \quad x = b(r)s + c(r)$$

$$y_s = a(r) \Rightarrow \quad y = a(r)s + d(r)$$

$$u_s = 2 \Rightarrow \quad u = 2s + e(r)$$

} cannot get rid of 5 arb functions w/ 3 eq^n's

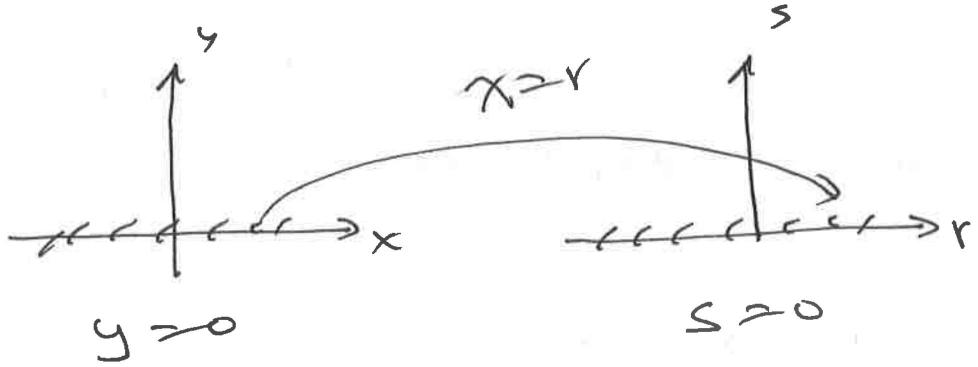
Could
$$x = qs + c$$
$$y = ps + d$$
} but we're bringing back in p & q

So if we could explicit find a, b, c, d & e then we have a better chance of finding u in terms of x & y. So we need to bring in the BC now!

so with

$$x_s = q, \; y_s = p, \; u_s = 2 \quad p_s = 0, \quad q_s = 0$$

we need $\underline{5}$ bc



so as before

on $s = 0$ $x = r,\ y = 0,\ u = x = r$

we need something for p & q

call the fone is $u(x, 0) = x$

so differentiate this wrt x $u_x(x, 0) = 1$

so on $s = 0$ $p = 1$

what about q - go to PDE $pq = 1$

so on $s = 0$ $1 \cdot q = 1 \Rightarrow q = 1$

Now we solve

	$s = 0$
$x_s = q$	$x = r$
$y_s = p$	$y = 0$
$u_s = 2$	$u = r$
$p_s = 0$	$p = 1$
$q_s = 0$	$q = 1$

(1) $p_s = 0 \Rightarrow p = a(v)$

$s = 0 \quad p = 1 \Rightarrow a = 1 \Rightarrow \boxed{p = 1}$ (always)

(2) $q_s = 0 \Rightarrow q = b(v)$

$s = 0 \quad q = 1 \Rightarrow b = 1$ so $\boxed{q = 1}$ (always)

(3) $u_s = 2 \Rightarrow u = 2s + c(v)$

$s = 0 \quad u = r \Rightarrow c = r \Rightarrow \boxed{u = 2s + r}$ (*)

(4) $x_s = q = 1 \Rightarrow x = s + d(v)$

$s = 0 \quad x = r \Rightarrow d(v) = r \Rightarrow \boxed{x = s + r}$ (*)

(5) $y_s = p = 1 \Rightarrow y = s + e(v)$

$s = 0 \quad y = 0 \Rightarrow e(v) = 0$ so $\boxed{y = s}$ (*)

Now eliminate r & s in the * eq^{n}s

$s = y \quad r = x - y$ so $u = 2y + x - y$

$u = x + y \Leftarrow$ the sol^n