Toronto Math Circles: Full Solutions Mathematics Competition

Junior: Solutions

Saturday December 5, 2015

1:00 pm - 3:00 pm

Each problem is graded on a basis of 0 to 10 points. All the necessary work to justify an answer must be shown clearly to obtain full credit. Some partial credit may be given, but only when a contestant has shown significant and substantial progress toward a solution.

1. A die is said to be regular if the number on its faces are $1, 2, 3, 4, 5, 6$. A die is said to be fair if the probability of landing on any face is the same. Two regular fair dice are rolled. What is the most likely greatest common divisor of the two numbers? What is the probability of this occurring? Note: Greatest common divisor is the same thing as greatest common factor.

   Answer. The most likely greatest common divisor is 1. This has probability $\frac{23}{36}$.
   Solution. To see this, simply write out the greatest common divisor table for the two dice

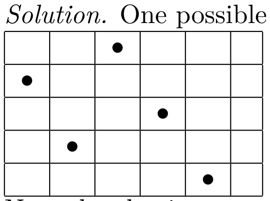
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

   Observe that 1 is the number that occurs the most. It occurs 23 out of 36 times. Therefore, the probability is $\frac{23}{36}$.

   Remark. What would the answer be if greatest common divisor is replaced with least common multiple?

2. Determine the maximum number of dots that can be placed in a 5×6 grid such that no two dots are in the same row, column or diagonal. Note: 5×6 means 5 rows and 6 columns.

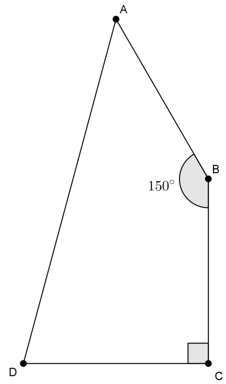
   Answer. The maximum number of dots is 5.
   Solution. One possible arrangement is

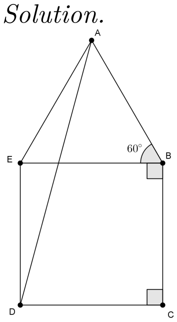
   Note that having anymore dots is not possible because otherwise two dots must be in the same row.

   Remark 1. If a question asks for the maximum, it is required to justify why there cannot be more.
   Remark 2. For a grid of size $m \times n$ where $4 \leq m \leq n$, the maximum number of dots such that no two dots are in the same row, column or diagonal is always m.

3. Let $ABCD$ be a quadrilateral such that $AB = BC = CD$ and $\angle ABC = 150°$ and $\angle BCD = 90°$. Determine the value of $\angle CDA$.



Answer. $\angle CDA = 75°$
Solution.



Let E be the point such that $BCDE$ forms a square. Thus, $\angle ABE = 60°$. Since $AB = BE$, then

$$\angle EAB = \angle AEB = 60°$$

Therefore, ABE is an equilateral triangle. Thus,

$$\angle DEA = \angle DEB + \angle BEA = 90° + 60° = 150°$$

Since $AE = ED$, then

$$\angle ADE = \frac{1}{2}(180° - 150°) = 15°$$

Therefore,

$$\angle CDA = \angle CDE - \angle ADE = 90° - 15° = 75°$$

Remark. Although there might be other ways to solve this problem such as using trigonometry, this solution is one of the easiest.

4. Consider the list of numbers

$$1, 12, 123, 1234, \ldots, 123456789, 1234567890, 12345678901, 123456789012, \ldots$$

Find the position of all number in this list that are divisible by 90. Note: 1 is at position 1 and 12 is at position 2 and etc.

Answer. $10n$ where n is a positive integer or $10, 20, 30, 40, \ldots$
Solution. For a number to be divisible by 90, it must be divisible by 10 and 9. For a number to be divisible by 10, it must end in a 0. Therefore, the numbers to consider are

$$1234567890, 12345678901234567890, 123456789012345678901234567890, \ldots (*)$$

For a number to be divisible by 9, the sum of it's digits must be divisible by 9. Since

$$\begin{aligned}1+2+3+4+5+6+7+8+9+0 &= (0+9)+(1+8)+(2+7)+(3+6)+(4+5)\\ &= 9+9+9+9+9\\ &= 45\end{aligned}$$

then the sum of the digits of each of the numbers in $(*)$ is

$$45, 2(45), 3(45), 4(45), \ldots$$

Each other these numbers is divisible by 9. Therefore, everything number in $(*)$ is divisible by 90. It remains to note that the position of these numbers are $10, 20, 30, 40, \ldots$.

5. Let x and y be two real numbers that satisfy the equation

$$x^2 + y^2 + 12xy = 2016$$

Determine the maximum possible value of xy and the values of x and y that makes this maximum possible.

Answer. Maximum of xy is 144. This occurs when $x = y = 12$ or $x = y = -12$.
Solution. First, add $2xy$ and subtract $2xy$ to the left hand side to get

$$\begin{aligned}x^2 + y^2 + 12xy &= x^2 - 2xy + y^2 + 2xy + 12xy\\ &= (x-y)^2 + 14xy\\ &= 2016\end{aligned}$$

Rearranging this gives

$$xy = \frac{1}{14}\left[2016 - (x-y)^2\right] = 144 - \frac{1}{12}(x-y)^2$$

The problem of maximizing xy is equivalent to minimizing $(x-y)^2$, which occurs when $x = y$. Therefore, the maximum of xy is 144. This is obtained when $x = y = 12$ or $x = y = -12$.

Remark 1. An alternate solution would be to use arithmetic-geometric mean inequality.
Remark 2. Another possible alternative solution would be to start with the result and show that every other (x, y) will produce a suboptimal result.

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Find the position of all number in this list that are divisible by 90.
Note: 1 is at position 1 and 12 is at position 2 and etc.

Answer. $10n$ where n is a positive integer
Solution. For a number to be divisible by 90, it must be divisible by 10 and 9. For a number to be divisible by 10, it must end in a 0. Therefore, the numbers to consider are

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For a number to be divisible by 9, the sum of it's digits must be divisible by 9. Since

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2. Compute the area of the region

$$\{(x,y) \in \mathbb{R}^2 | (x^2+y^2)^2 \leq 4x^2, |x|+|y| \geq 1\}$$

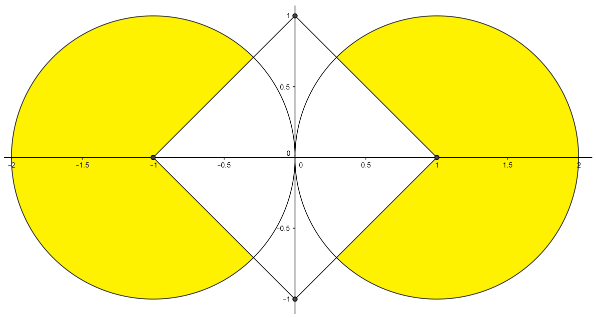
Answer. $\frac{3\pi}{2}$
Solution. When $x \geq 0$, $(x^2+y^2)^2 \leq 4x^2$ is equivalent to $x^2+y^2 \leq 2x$ which is equivalent to $(x-1)^2+y^2 \leq 1$. When $x \leq 0$, $(x^2+y^2)^2 \leq 4x^2$ is equivalent to $x^2+y^2 \leq -2x$ which is equivalent to $(x+1)^2+y^2 \leq 1$. Therefore,

$$(x^2+y^2)^2 \leq 4x^2$$

represents the region contained inside the two circles whose centers are at $(1,0)$ and $(-1,0)$ with radii 1.
When $x \geq 0$ and $y \geq 0$, $|x|+|y| \geq 1$ is equivalent to $x+y \geq 1$, which is the unbounded region that is separated by $x+y=1$. Repeating this with the other four quadrants shows that

$$|x|+|y| \geq 1$$

represents the region outside of the square whose vertices are at $(1,0), (0,1), (-1,0), (0,-1)$.
Combining these two regions gives the diagram



The sectors that are removed has angle 90° each. Therefore, the area of the shaded region is

$$2\left(\frac{3}{4}\right)\pi = \frac{3\pi}{2}$$

Remark. The region was designed with PAC-MAN in mind.

3. Let x_n be a sequence of real numbers such that $x_1 = 1$ and $0 \le x_k \le 2x_{k-1}$ for $k = 2, 3, \ldots$. Let m be a non-negative integer, determine the maximum value of

$$x_1 - x_2 + x_3 - x_4 + \cdots + x_{2m+1} - x_{2m+2}$$

Express your answer in simplest form.

Answer. $\frac{1}{3}\left(2^{2m+1}+1\right)$ and equality occurs when $x_k = 2^{k-1}$ for $k = 1, 2, \ldots, 2m+1$ and $x_{2m+2} = 0$.
Solution. When $m = 0$ then

$$x_1 - x_2 \le x_1 = 1$$

and equality occurs when $x_1 = 1$ and $x_2 = 0$.
When $m = 1$ then

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= x_1 + (x_3 - x_2) - x_4 \\ &\le x_1 + (x_3 - x_2) \\ &\le x_1 + x_2 \\ &\le 1 + 2 \\ &= 3 \end{aligned}$$

and equality occurs when $x_1 = 1, x_2 = 2, x_3 = 4$ and $x_4 = 0$.
When $m \ge 2$,

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 + \cdots + x_{2m+1} - x_{2m+2} &= x_1 + (x_3 - x_2) + \cdots + (x_{2m+1} - x_{2m}) - x_{2m+2} \\ &\le x_1 + (x_3 - x_2) + \cdots + (x_{2m+1} - x_{2m}) \\ &\le x_1 + x_2 + x_4 + \cdots + x_{2m} \\ &\le 1 + 2 + 2^3 + \cdots + 2^{2m-1} \\ &= 1 + \sum_{n=1}^{m} 2^{2n-1} \\ &= 1 + 2\sum_{n=0}^{m-1} 2^{2n} \\ &= 1 + 2\sum_{n=0}^{m-1} 4^n \\ &= 1 + 2\left(\frac{4^m - 1}{4 - 1}\right) \\ &= \frac{1}{3}\left(2^{2m+1} + 1\right) \end{aligned}$$

Equality occurs when $x_k = 2^{k-1}$ for $k = 1, 2, \ldots, 2m+1$ and $x_{2m+2} = 0$. Note that this generalization also covers the $m = 0$ and $m = 1$ cases.

Remark. The two extra split cases are required because the second equal sign in $m \geq 2$ case does not make sense for $m = 0$ and $m = 1$.

4. A die is said to be fair if the probability of landing on any face is the same. Let A and B be two 6 sided fair dice with a positive integer on each face. Suppose that the sum of their rolls gives the following probability table

Sum: $(n =)$	2	3	4	5	6	7	8	9	10	11	12
Probability of n	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Find all possible combinations of dice, A and B, such that the set of numbers on die A is different from the set of numbers on die B.
Note: This is the same table for when A and B are both standard regular die.

Answer. Die A: $(1, 2, 2, 3, 3, 4)$ Die B: $(1, 3, 4, 5, 6, 8)$
Solution. Consider a regular fair die with numbers $(1, 2, 3, 4, 5, 6)$. This can be encoded through generating functions

$$x + x^2 + x^3 + x^4 + x^5 + x^6$$

Note that the exponent represent the dice number and the coefficient represent which contains that dice number. The sum of two dice rolls can be expressed as

$$\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2$$

Expanding this and dividing by 36 will give precisely the table in the question. Therefore, the goal is the determine all polynomials $f(x)$ and $g(x)$ such that
(i) $fg = \left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2$
(ii) The coefficients of f and g are non-negative
(iii) The sum of the coefficients of each of f and g is 6, $f(1) = g(1) = 6$
(iv) f and g does not have a constant term
First, factor the original polynomial into irreducible functions

$$\begin{aligned}\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2 &= x^2\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^2 \\ &= \frac{x^2\left(1 - x^6\right)^2}{(1-x)^2} \\ &= \frac{x^2\left(1 - x^3\right)^2\left(1 + x^3\right)^2}{(1-x)^2} \\ &= x^2\left(1 + x + x^2\right)^2(1+x)^2\left(1 - x + x^2\right)^2\end{aligned}$$

To satisfy (iv), x^2 must be split into both f and g. Next, evaluating the three polynomials

$$1 + x + x^2, 1 + x, 1 - x + x^2$$

at $x = 1$ yields $3, 2, 1$, respectively. Thus, to satisfy (iii), both the polynomials $\left(1 + x + x^2\right)^2$ and $(1+x)^2$ must be split into both f and g. Finally, there are two ways to handing $\left(1 - x + x^2\right)^2$. The first ways is to split them into both f and g, but this will result in die A and B having the same set of numbers. The second way is to give both terms to $f(x)$. Combining all this gives

$$\begin{aligned}f(x) &= x\left(1 + x + x^2\right)(1+x)\left(1 - x + x^2\right)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8 \\ g(x) &= x\left(1 + x + x^2\right)(1+x) = x + 2x^2 + 2x^3 + x^4\end{aligned}$$

Note that they both satisfy (ii). Therefore, there is only one possible combination of dice, A and B, such that the set of numbers on die A is different from the set of numbers on die B. Die A:

$(1, 2, 2, 3, 3, 4)$ Die B: $(1, 3, 4, 5, 6, 8)$.

Remark 1. An alternative solution would be to build up both dices starting with “they must both have at least one 1”.
Remark 2. The dice in the solution is actually called the Sicherman dice.

5. Let x_n be a sequence of positive real numbers such that

$$x_n = \sqrt{\frac{6x_{n-1}^3 x_{n-3} - 8x_{n-2}^3 x_{n-1}}{x_{n-2}x_{n-3}}}$$

and $x_1^2 = 1$, $x_2^2 = 2$ and $x_3^2 = 12\sqrt{2}$. Prove that for any positive integer n,

$$x_n \prod_{k=1}^{n} x_k$$

is an integer and divisible by n.

Solution. When $n = 1$ the result is obviously true. For the rest of this problem, assume $n \geq 2$. The original relation can be rewritten as

$$\frac{x_n^2}{x_{n-1}} = 6\frac{x_{n-1}^2}{x_{n-2}} - 8\frac{x_{n-2}^2}{x_{n-3}}$$

Define $y_n = \frac{x_n^2}{x_{n-1}}$ then $y_2 = 2$, $y_3 = 12$, and

$$y_n = 6y_{n-1} - 8y_{n-2}$$

for every $n \geq 4$. The characteristic polynomial for this recurrence relation is

$$y^2 - 6y + 8 = (y-2)(y-4) = 0$$

Therefore, the solution to the recurrence relation is

$$y_n = A\left(2^n\right) + B\left(4^n\right)$$

for some A and B. Setting $n = 2$ and $n = 3$ gives the system of equations

$$\begin{cases} 4A + 16B = 2 \\ 8A + 64B = 12 \end{cases}$$

Solving this system gives $A = -\frac{1}{2}$ and $B = \frac{1}{4}$. Therefore, $y_n = 4^{n-1} - 2^{n-1} = 2^{n-1}\left(2^{n-1} - 1\right)$. Observe that

$$\begin{aligned} x_n \prod_{k=1}^{n} x_k &= \prod_{k=2}^{n} y_k \\ &= \prod_{k=2}^{n} 2^{k-1}\left(2^{k-1} - 1\right) \\ &= \prod_{k=1}^{n-1} 2^k\left(2^k - 1\right) \\ &= 2^{\sum_{k=1}^{n-1} k} \prod_{k=1}^{n-1}\left(2^k - 1\right) \\ &= 2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n-1}\left(2^k - 1\right) \end{aligned}$$

To see that $x_n \prod_{k=1}^{n} x_k$ is an integer for every n, simply note that each y_k is generated from a linear integer combination of the previous terms. To see that $x_n \prod_{k=1}^{n} x_k$ is divisible by n, let $n = 2^l m$ where m is an odd number and l is a non-negative integer. Observe that

$$l \leq n - 1 \leq \frac{n(n-1)}{2}$$

Thus,

$$2^l | 2^{\frac{n(n-1)}{2}}$$

Next, since m is an odd number, then $\gcd(2, m) = 1$. Therefore,

$$2^{\phi(m)} \equiv 1 \mod m$$

where ϕ is the Euler Quotient function. Since $\phi(m) \leq m - 1 \leq n - 1$ then one of

$$2^1 - 1, 2^2 - 1, \ldots, 2^{n-1} - 1$$

is divisible by m. Therefore, $x_n \prod_{k=1}^{n} x_k$ is divisible by n.

Remark 1. The separate case for $n = 1$ is required because $\prod_{k=2}^{n} y_k$ does not make sense for $n = 1$.
Remark 2. Alternative method for solving for y_n is through generating functions.
Remark 3. Instead of showing how to actually solve for y_n, it is possible to just claim the result and prove it by induction.
Remark 4. It is possible to solve this problem without appealing to Euler Quotient function. One possibility is to show that every prime power that divides n also divides x_n. This will require Fermat's Little Theorem.
Remark 5. It might be possible to use induction to solve this problem.